Regina Gente

The Varchenko Matrix for Cones

Philipps Universität Marburg

The Varchenko Matrix for Cones

Dissertation
zur
Erlangung des Doktorgrades
der Naturwissenschaften
(Dr. rer. nat.)

dem
Fachbereich Mathematik und Informatik
der Philipps-Universität Marburg
vorgelegt von

Regina Gente
geboren in Gifhorn

Marburg im Juli 2013

Philipps Universität Marburg

Erstgutachter:	Prof. Dr. V. Welker
Zweitgutachter:	Prof. Dr. I. Heckenberger
Einreichungstermin:	4. Juli 2013
Prüfungstermin:	19. September 2013
Hochschulkennziffer:	1180

Zusammenfassung

Das Hauptziel dieser Arbeit ist es, den bekannten Satz von Varchenko [19] über Arrangements von Hyperebenen auf eine allgemeinere Situation zu erweitern und den ersten vollständigen und elementaren Beweis des Satzes von Varchenko zu liefern.

Zunächst stellen wir den Satz von Varchenko kurz dar. Hierfür definieren wir zuerst die wichtigsten Begriffe. Wir betrachten im $\mathbb{R}^n$ ein endliches Arrangement von Hyperebenen, welches wir mit $\mathcal{A}$ bezeichnen. Jeder Hyperebene $H \in \mathcal{A}$ weisen wir ein Gewicht a_H zu, das als Unbekannte im Ring $R_{\mathcal{A}} := \mathbb{Z}[a_H, H \in \mathcal{A}]$ betrachtet wird. Die Gebiete des Arrangements $\mathcal{A}$ sind die offenen Zusammenhangskomponenten des Komplements $\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{A}} H$. Wir bezeichnen die Menge aller Gebiete mit $\mathcal{P}(\mathcal{A})$. Sei M der von den Gebieten in $\mathcal{P}(\mathcal{A})$ frei erzeugte $R_{\mathcal{A}}$-Modul. Varchenko definiert auf dem Modul M eine symmetrische $R_{\mathcal{A}}$-Bilinearform $B(\cdot,\cdot)$ durch

$$B(P,Q) = \prod_{H \in T} a_H,\ T := \{H \in \mathcal{A} \mid H \text{ trennt } P \text{ und } Q\}$$

für $P, Q \in \mathcal{P}(\mathcal{A})$. Wir bezeichnen die Menge der Schnitte $\{\bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A} \text{ so dass } \bigcap_{H \in \mathcal{B}} H \neq \emptyset\}$ des Arrangements $\mathcal{A}$ mit $\mathcal{E}(\mathcal{A})$. Das Gewicht eines Schnittes $L \in \mathcal{E}(\mathcal{A})$ definieren wir durch

$$a(L) = \prod_{H \in \mathcal{A} : L \subseteq H} a_H \ .$$

Jedem Schnitt $L \in \mathcal{E}(\mathcal{A})$ weisen wir eine natürliche Zahl wie folgt zu: Sei $H \in \mathcal{A}$ eine Hyperebene, so dass $L \subseteq H$. Dann ist $2 \cdot l_{\mathcal{P}(\mathcal{A})}(L)$ gleich der Anzahl der Gebiete $P \in \mathcal{P}(\mathcal{A})$, die die Eigenschaft haben, dass L der minimale Schnitt ist, der $\bar{P} \cap H$ enthält.

Nun sind wir in der Lage den Satz von Varchenko zu formulieren:

Satz (Satz von Varchenko)**.** *Sei $\mathcal{A}$ ein Arrangement von Hyperebenen. Dann gilt für die Determinante der Bilinearform $B(\cdot,\cdot)$*

$$\det B(\mathcal{A}) = \prod_{L \in \mathcal{E}(\mathcal{A})} (1 - a(L)^2)^{l_{\mathcal{P}(\mathcal{A})}(L)} \ .$$

Der Beweis von Varchenko ist in [19] zu finden. Weiterführende Literatur, auf die Varchenko für seinen Beweis zurückgreift, sind seine Veröffentlichungen [18] und [20]. Für die Definition der Exponenten in der Faktorisierung der Determinante verwendet Varchenko selbst zunächst Methoden aus der projektiven Geometrie. Die Definition, welche oben verwendet wird, findet sich implizit im Verlauf des von Varchenko in [19] geführten Beweises und wird von Denham und Hanlon in

[7] verwendet.
Es ist anerkannt, dass Varchenko den ersten Beweis des obigen Satzes lieferte. Dieser Beweis bereitet wegen der verwendeten Notation bezüglich der Lesbarkeit größere Schwierigkeiten und Varchenkos Beweisführungen und Argumentationen werden häufig diskutiert.
Der in dieser Arbeit geführte Beweis ist unabhängig von Varchenkos Beweisführung. In Teilen wird ein Beweisversuch von Denham und Hanlon für den Spezialfall eines zentralen Arrangements (siehe [7]) verwendet. Dieser Beweisversuch weist jedoch Lücken auf und ist zumindest an einer Stelle fehlerhaft. Die Lücken werden hier geschlossen und die Argumentation so geändert und ergänzt, dass der Spezialfall gezeigt werden kann. Anschließend wird dieser genutzt um die Verallgemeinerung des Satzes von Varchenko zu beweisen.

Aus der von Varchenko gefundenen Formel für die Determinante der Matrix $B(\cdot,\cdot)$ können wir ablesen, dass die Determinante durch die Kombinatorik des Arrangements bestimmt ist und sich elegant faktorisieren lässt. Desweiteren erlaubt die Faktorisierung der Determinante der Bilinearform eine strukturelle Untersuchung der zugehörigen Matrix. Aus der Faktorisierung können wir sehr einfach die Werte der Paramter, für die der Kern der Matrix $B(\mathcal{A})$ nicht trivial ist, ablesen.
Diese sind von besonderem Interesse, wenn $\mathcal{A}$ das Spiegelungsarrangement der symmetrischen Gruppe ist ([17]) und die zugehörigen Kerne werden in diesem Fall für die Darstellungstheorie von gewissen quantisierten Kac-Moody Lie Algebren verwendet.
Eine Erweiterung diese Zusammenhanges auf Arrangements im Komplexen ist in [16] zu finden.

In dieser Arbeit betrachten wir nun die folgende allgemeinere Situation: Sei $\mathcal{C}$ ein zentrales Teilarrangement von $\mathcal{A}$ und $\mathcal{K}$ ein Gebiet von $\mathcal{C}$. Wir nennen $\mathcal{K}$ einen Kegel des Arrangements $\mathcal{A}$ und schränken die Bilinearform $B(\cdot,\cdot)$ auf die Gebiete aus $\mathcal{P}(\mathcal{A})$, die in $\mathcal{K}$ liegen, ein. Die eingeschränkte Menge bezeichnen wir mit $\mathcal{P}_{\mathcal{K}}(\mathcal{A})$. Mit $\mathcal{E}_{\mathcal{K}}(\mathcal{A})$ bezeichnen wir die Menge der Schnitte, für die gilt $L \cap \mathcal{K} \neq \emptyset$. Wir weisen erneut jedem Schnitt $L \in \mathcal{E}_{\mathcal{K}}(\mathcal{A})$ eine natürliche Zahl zu: Sei $H \in \mathcal{A}$ eine Hyperebene, so dass $L \subseteq H$. Dann ist $2 \cdot l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(L)$ die Anzahl der Gebiete aus $\mathcal{P}_{\mathcal{K}}(\mathcal{A})$, so dass L der minimale Schnitt ist, der $\bar{P} \cap H$ enthält.
Als Hauptresultat zeigen wir, dass die Determinante dieser eingeschränkten Matrix durch die Kombinatorik des Arrangements $\mathcal{A}$ innerhalb des Kegels $\mathcal{K}$ bestimmt ist und sich elegant faktorisieren lässt:

Satz. *(Varchenkos Satz für Kegel) Sei $\mathcal{A}$ ein Arrangement und $\mathcal{K}$ ein Kegel des Arrangements $\mathcal{A}$. Dann gilt für die Determinante der auf den Kegel eingeschränkten Bilinearform:*

$$\det B_{\mathcal{K}}(\mathcal{A}) = \prod_{L \in \mathcal{E}_{\mathcal{K}}(\mathcal{A})} (1 - a(L)^2)^{l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A}}(L)} \ .$$

Im Falle des leeren Teilarrangements $\mathcal{C}$ folgt, dass der Kegel $\mathcal{K}$ gleich dem gesamten $\mathbb{R}^n$ ist. Folglich ist Varchenkos Satz ein Spezialfall des hier bewiesenen Satzes von Varchenko für Kegel (siehe Theorem 4.5).

Die vorliegende Arbeit ist wie folgt gegliedert: Nach der deutschen Zusammenfassung in diesem Abschnitt, wird die Arbeit englischsprachig fortgesetzt. Wir beginnen mit einer Einleitung in Kapitel 1 und stellen anschließend die benötigten Grundlagen in Kapitel 2 dar. In Kapitel 3 wird Varchenkos Satz für den Spezialfall eines zentralen Arrangements bewiesen (Theorem 3.5) und schließlich folgt der Beweis von Varchenkos Satz für Kegel (Theorem 4.5) in Kapitel 4. Zum Abschluss der Arbeit wird eine Anwendungs von Satz Theorem 4.5 in Kapitel 5 betrachtet.

Um den Hauptsatz dieser Arbeit (Theorem 4.5) zu beweisen, zeigen wir zunächst in Kapitel 3 den Spezialfall des Satzes von Varchenko für zentrale Arrangements (Theorem 3.5).
Wir betrachten das reduzierte Edelman Poset (siehe [9] für Details zum Edelman Poset und Kapitel 3.2) und zeigen, dass dessen Möbius Funktion über die reduzierte Eulercharakteristik eines polyedrischen Komplexes bestimmt werden kann (siehe Theorem 3.13). Diese Aussage ist auch in [7] zu finden. Der dort angekündigte Beweis ist jedoch nicht veröffentlicht worden.
Mit Hilfe der Möbius Funktion des reduzierten Edelman Posets kann eine Faktorisierung der Matrix $B(\mathcal{A})$ für ein zentrales Arrangement $\mathcal{A}$ formuliert werden (siehe Theorem 3.35). Dieses Theorem und die für seinen Beweis benötigten Aussagen sind auch in [7] zu finden. Wir passen diese auf die hier betrachtete Situation gegebenenfalls an und geben ausführliche Beweise.
In Theorem 3.17 geben wir im ersten Teil eine Formel für die Berechnung der Möbius Funktion in einem Spezialfall an und sagen im zweiten Teil aus, dass unter bestimmten Umständen die Möbiusfunktion den Wert 0 annimmt. In [7] ist eine Proposition zu finden, die dem erste Teil von Theorem 3.17 ähnelt. In der dortigen Form ist diese Proposition leider nicht korrekt und der angekündigte Beweis ist nie erschienen.
Unter Verwendung von Theorem 3.17 und geometrischen Argumenten (siehe hierzu Abschnitt 3.4) können wir anschließend zeigen, dass sich jede der in der Faktorisierung aus Theorem 3.35 auftretenden Matrizen bei geeigneter Anordnung der Gebiete, welche Zeilen und Spalten indizieren, als untere Dreiecksmatrix mit Blockstruktur darstellen lässt. Hierüber können wir eine Faktorisierung der Determinante von $B(\mathcal{A})$ bestimmen und somit Theorem 3.5 beweisen.
Wir beenden Kapitel 3 mit einem detaillierten Beispiel.

In Kapitel 4 zeigen wir Theorem 4.5 unter der Verwendung von Theorem 3.5. Hierbei genügt es, wenn wir den Fall betrachten, dass $\mathcal{K}$ ein Kegel eines zentralen Arrangements $\mathcal{A}$ ist: Mittels Coning und Deconing können wir ein (affines) Arrange-

ment im $\mathbb{R}^n$ in ein zentrales Arrangement im $\mathbb{R}^{n+1}$ überführen und umgekehrt. Die Kombinatorik des Arrangements innerhalb des Kegels $\mathcal{K}$ wird von diesen Transformationen nicht beeinflusst.
Wir beweisen zunächst algebraische Eigenschaften der Determinante der Matrix $B_{\mathcal{K}}(\mathcal{A})$ (siehe Lemma 4.6) und der Terme $1 - a(L)^2$ für $L \in \mathcal{E}_{\mathcal{K}}(\mathcal{A})$ (siehe Corollary 4.8). Wir zeigen anschließend ein Lemma (Lemma 4.10), welches hilfreiche Informationen zu den Gebieten, die zur Bestimmung der Exponenten in der Faktorisierung gezählt werden, liefert.
Dadurch, dass wir die Gewichte der Hyperebenen des die Kegel definierenden Teilarrangements $\mathcal{C}$ gleich 0 setzen, erhalten wir eine Blockmatrix, in der jeder Block einer Matrix $B_{\mathcal{K}}(\mathcal{A})$ für $\mathcal{K} \in \mathcal{P}(\mathcal{C})$ entspricht. Hiermit und unter Verwendung von algebraischen Methoden erhalten wir, dass $\det B(\mathcal{A})$ von $\det B_{\mathcal{K}}(\mathcal{A})$ geteilt wird. Im nächsten Schritt bestimmen wir die Exponenten der in $\det B_{\mathcal{K}}(\mathcal{A})$ auftretenden Faktoren. Hierbei können wir uns auf die zu Schnitten $L \in \mathcal{E}_{\mathcal{K}}(\mathcal{A})$ auftretenden Faktoren beschränken. Hierfür nutzen wir erneut, dass die Matrizen $B_{\mathcal{K}}(\mathcal{A})$ bei geeigneter Wahl der Gewichte Blockstruktur erhalten, und verwenden Abschätzungen in Kombination mit den Ergebnissen aus Lemma 4.10. Zuletzt wird das Vorzeichen von $\det B_{\mathcal{K}}(\mathcal{A})$ bestimmt, indem wir die zugehörige Matrix betrachten, wenn alle Gewichte gleich 0 gesetzt sind.
Auch in diesem Kapitel wird am Ende ein ausführliches Beispiel dargestellt.

Im Falle des Spiegelungsarrangements der symmetrischen Gruppe wurde die Determinante der Bilinearform von Zagier [22] unabhängig von Varchenko bestimmt. Zagier betrachtet die Hyperebenen $H_{i,j} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = x_j\}$ und das von ihnen gebildete Arrangement $\mathcal{B}_n := \{H_{i,j} \mid 1 \leq i < j \leq n\}$. Die Gewichte der Hyperebenen werden alle gleich q gesetzt. In Kapitel 5 betrachten wir nun nicht nur ausschließlich die gesamte Menge der Gebiete des Arrangements $\mathcal{B}_n$. Es lässt sich zu der Menge der linearen Erweiterungen einer partiell geordneten Menge P jeweils ein Kegel $\mathcal{K}$ finden, so dass die Menge der linearen Erweiterungen in Bijektion zur Menge der Gebiete in $\mathcal{K}$, $\mathcal{P}_{\mathcal{K}}(\mathcal{B}_n)$, steht. Dies ist die Grundlage dafür, dass wir Theorem 4.5 anwenden können. In Theorem 5.6 geben wir schließlich eine kombinatorische Beschreibung der Determinante der auf $\mathcal{K}$ eingeschränkten Bilinearform, die Invarianten der partiell geordneten Menge P und sich aus P ergebenden partiell geordneten Mengen verwendet.

Contents

1 Introduction

The main aim of this thesis is to extend a famous theorem of Varchenko [19] on hyperplane arrangements to a more general situation and to give the first complete and elementary proof of Varchenko's Theorem.

First, we briefly introduce Varchenko's Theorem. We consider an arrangement of hyperplanes $\mathcal{A}$ in $\mathbb{R}^n$. To each hyperplane $H \in \mathcal{A}$ we attach a weight a_H, which is treated as an indeterminate in the ring $R_{\mathcal{A}} := \mathbb{Z}[a_H, H \in \mathcal{A}]$. The regions of the arrangement $\mathcal{A}$ are the open components of the complement $\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{A}} H$. We denote the set of regions by $\mathcal{P}(\mathcal{A})$. Let M be the $R_{\mathcal{A}}$-module freely generated by the regions in $\mathcal{P}(\mathcal{A})$. Varchenko defined a symmetric $R_{\mathcal{A}}$-bilinear form $B(\cdot,\cdot)$ the module M by setting

$$B(P,Q) = \prod_{H \in T} a_H,\ T := \{H \in \mathcal{A} \mid H \text{ seperates } P \text{ and } Q\}$$

for $P, Q \in \mathcal{P}(\mathcal{A})$. We denote the set of intersections $\{\bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A}$ such that $\bigcap_{H \in \mathcal{B}} H \neq \emptyset\}$ of the arrangement $\mathcal{A}$ by $\mathcal{E}(\mathcal{A})$. We define the weight of an intersection $L \in \mathcal{E}(\mathcal{A})$ by $a(L) = \prod_{H \in \mathcal{A}:\ L \subseteq H} a_H$. To each intersection $L \in \mathcal{E}(\mathcal{A})$ we assign a natural number the following way: Let $H \in \mathcal{A}$ be a hyperplane such that $L \subseteq H$. Then $2 \cdot l_{\mathcal{P}(\mathcal{A})}(L)$ equals the number of regions $P \in \mathcal{P}(\mathcal{A})$, which have the property that L is the minimal (with respect to inclusion) intersection containing $\bar{P} \cap H$.
Now everything is defined for recalling Varchenko's Theorem (see [19]):

Theorem (Varchenko's Theorem). *Let $\mathcal{A}$ be an arrangement of hyperplanes. Then for the determinant of the bilinear form $B(\cdot,\cdot)$ the following holds:*

$$\det B(\mathcal{A}) = \prod_{L \in \mathcal{E}(\mathcal{A})} (1 - a(L)^2)^{l_{\mathcal{P}(\mathcal{A})}(L)} .$$

For Varchenko's proof of this theorem see [19]. Further details which he has used for his proof can be found in [20] and [18]. Concerning the exponents in the factorisation of the determinant, Varchenko himself uses projective geometry, when he formulates his theorem. The version of this result described above is stated implicitly later in his paper or can be found in [7]. Although Varchenko was the first who proved this theorem, his proof has some deficits concerning readability and his argumentation has been widely discussed in the community.
The proof given in this thesis is independent from Varchenko's argumentation. In [7] Denham and Hanlon try to give a proof for the special case of a central arrangement. There are some gaps and their argumentation is not always correct. Here, we fill in the gaps, amend and extend the argumentation to show this special case. Subsequent this special case will be used for proving a more general version of Varchenko's theorem.

By Varchenko's Theorem it holds, that the determinant of the matrix of $B(\cdot,\cdot)$ is determined by the combinatorics of the arrangement and allows a nice factorisation. Furthermore Varchenko's formula for the determinant allows a structural analysis concerning the properties of the matrix. The values for which the determinant vanishes can be read of easily from the factorisation. Therefore, the values of the parameters for which the matrix $B(\mathcal{A})$ has a non-trivial null-space are characterized by Varchenko's formula.
This relation is particularly important in the case if $\mathcal{A}$ is the reflection arrangement of the symmetric group (see [17]). In this case the null-spaces play a role in the representation theory for quantum Kac-Moody Lie algebras.
For an extension of this result to the arrangement consisting of all diagonal hyperplanes in a complex space see [16].

In this thesis we consider the following more general situation: Let $\mathcal{C}$ be a central subarrangement of $\mathcal{A}$ and let $\mathcal{K}$ be a region of this subarrangement. We call $\mathcal{K}$ a cone of the arrangement $\mathcal{A}$ and restrict the bilinear form $B(\cdot,\cdot)$ to the regions from $\mathcal{P}(\mathcal{A})$ that lie in $\mathcal{K}$. We denote this set of regions by $\mathcal{P}_{\mathcal{K}}(\mathcal{A})$. By $\mathcal{E}_{\mathcal{K}}(\mathcal{A})$ we denote the set of intersections such that $L \cap \mathcal{K} \neq \emptyset$. As above, to each intersection $L \in \mathcal{E}_{\mathcal{K}}(\mathcal{A})$ we assign a natural number: Let $H \in \mathcal{A}$ be a hyperplane such that $L \subseteq H$. Then $2 \cdot l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(L)$ equals the number of regions $P \in \mathcal{P}_{\mathcal{K}}(\mathcal{A})$, which have the property that L is the minimal (with respect to inclusion) intersection containing $\bar{P} \cap H$.
We show that the determinant of this restricted matrix is determined by the combinatorics of the arrangement $\mathcal{A}$ inside the cone $\mathcal{K}$ and factors nicely (see Theorem 4.5):

Theorem (Varchenko's Theorem for Cones)**.** *Let $\mathcal{A}$ be an arrangement and $\mathcal{K}$ a cone of $\mathcal{A}$. Then for the determinant of the bilinear form restricted to the cone the following holds:*

$$\det B_{\mathcal{K}}(\mathcal{A}) = \prod_{L \in \mathcal{E}_{\mathcal{K}}(\mathcal{A})} (1 - a(L)^2)^{l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(L)} .$$

Since Varchenko's Theorem equals the situation of the empty subarrangement $\mathcal{C}$, we conclude that his theorem is a special case of Theorem 4.5.

The thesis is structured as follows: We start with an introduction in Chapter 1 and provide some basics in Chapter 2. In Chapter 3 we prove Varcheko's Theorem in the case of a central arrangement (Theorem 3.5) and subsequently we prove Varchenko's Theorem for Cones (Theorem 4.5) in Chapter 4. We finish the thesis by an application of Theorem 4.5 in Chapter 5.

In order to show that Theorem 4.5 holds, we prove Varchenko's Theorem in the case of a central arrangement first (see Theorem 3.5, Chapter 3). We consider the reduced Edelman poset (for the definition and details of the Edelman poset see

[9] and Chapter 3.2) and show that its Möbius function equals the reduced Euler Characteristic of some polyhedral complex (see Theorem 3.13). This theorem can be found in [7], too. There was a proof announced but it never appeared.
By using the Möbius function of the reduced Edelman poset, we show that there is a factorisation of the matrix $B(\mathcal{A})$ in the case that $\mathcal{A}$ is a central arrangement (see Theorem 3.35). This theorem and the lemmata needed for its proof can be found in [7]. If necessary, we adopt them to the considered situation, and we give detailed proofs for all of them.
In the first part Theorem 3.17 we give a formula for the Möbius function of the reduced Edelman poset in a special case. There is a proposition in [7], which resembles this part of the theorem. Unfortunately this proposition is not correct and the announced proof was never published. In the second part of Theorem 3.17 we show, that in another special case the Möbius function of the reduced Edelman poset equals 0.
By Theorem 3.17 and some geometric arguments (see section 3.4) we conclude that the matrices appearing in the factorisation of Theorem 3.35 are lower triangular blockmatrices. This allows us to determine a factorisation of the determinant of $B(\mathcal{A})$. Hence, Theorem 3.5 is proved. To illustrate Theorem 3.5 and its proof we provide a detailed example at the end of Chapter 3.

In Chapter 4 we deduce Theorem 4.5 from Theorem 3.5. We use the construction of coning and deconing, by which a central arrangement in $\mathbb{R}^n$ is transformed into an (affine) arrangement in $\mathbb{R}^{n-1}$ and vice versa. Hence it is sufficient to consider only central arrangements and their cones.
First, we show some algebraic properties of the determinant of $B_{\mathcal{K}}(\mathcal{A})$ (see Lemma 4.6) and of the terms $1 - a(L)^2$, $L \in \mathcal{E}_{\mathcal{K}}(\mathcal{A})$ (see Corollary 4.8). We prove a helpful lemma Lemma 4.10 concerning the regions which are counted to determine the exponent in the factorisation of the determinant.
Using the fact that we can make $B(\mathcal{A})$ into a block-matrix by setting the weights of the hyperplanes to zero which are defining the cones, and by some facts from algebra we get that $\det B_{\mathcal{K}}(\mathcal{A})$ divides $\det B(\mathcal{A})$ for every cone $\mathcal{K}$. We determine the exponent for every $L \in \mathcal{E}_{\mathcal{K}}(\mathcal{A})$ by converting the matrices $B_{\mathcal{K}}(\mathcal{A})$ by a change of weights again into block matrices and using some estimations in combination with Lemma 4.10. Finally we determine the sign of $\det B_{\mathcal{K}}(\mathcal{A})$ by considering the matrix with all weights set to 0.
Again we round off the section with an example.

In the case of the reflection arrangement of the symmetric group the determinant of the bilinear form has been determined by Zagier [22] independently from Varchenko's work. He considers the hyperplanes $H_{i,j} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = x_j\}$ in $\mathbb{R}^n$ and the arrangement $\mathcal{B}_n := \{H_{i,j} \mid 1 \le i < j \le n\}$. The weights of the $H_{i,j}$ are all set to q. Under a certain identification the linear extensions of a partially ordered set P are in bijection with the regions inside a cone of the reflection arrangement. In Chapter 5 we apply Theorem 4.5 in the described situation and give a combinatorial description of the determinant of the bilinear form in Theorem 5.6 by using some invariants of the partially ordered set P and partially ordered sets arising from it.

2 Basics

In this chapter we give a brief introduction to the theoretical basics of this thesis.

2.1 Arrangements of hyperplanes

Here, we present some basics concerning arrangements of hyperplanes. For general information on this subject see [13]. Throughout this thesis we are always working in the real space $\mathbb{R}^n$.

Definition 2.1. Let H be an $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$. Then H is called a hyperplane in $\mathbb{R}^n$.

Each hyperplane in $\mathbb{R}^n$ can be described by a polynomial $f : \mathbb{R}^n \to \mathbb{R}$ of degree 1 by

$$H := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid f(x) = 0\} \ .$$

Definition 2.2. Let $\mathcal{A}$ be a finite collection of hyperplanes in $\mathbb{R}^n$. Then $\mathcal{A}$ is called an (affine) arrangement of hyperplanes.

There are several types of arrangements of hyperplanes:

Definition 2.3. Let $\mathcal{A}$ be an arrangement of hyperplanes. $\mathcal{A}$ is called central, if $\bigcap_{H\in\mathcal{A}} H \neq \emptyset$. Furthermore, $\mathcal{A}$ is called essential, if $\bigcap_{H\in\mathcal{A}} H = \{0\}$.

Note that each essential arrangement is central, but not necessarily the other way round. If $\mathcal{A}$ is central, we can choose coordinates such that the hyperplanes in $\mathcal{A}$ are the kernels of linear functions $f : \mathbb{R}^n \to \mathbb{R}$. Therefore, we can make each arrangement with $\dim(\bigcap_{H\in\mathcal{A}} H) = 0$ into an essential one.

We consider the complement of the arrangement $\mathcal{A}$ in $\mathbb{R}^n$:

Definition 2.4. The open connected components of the complement $\mathbb{R}^n \backslash \bigcup_{H\in\mathcal{A}} H$ are called regions. We denote the set of all regions of the arrangement $\mathcal{A}$ by $\mathcal{P}(\mathcal{A})$.

Every region can be considered as a convex (open) polyhedron. If the region is bounded, it is actually a (open) polytope. We will define polyhedra and polytopes later on.

Definition 2.5. Let $\mathcal{B} \subseteq \mathcal{A}$ be a subarrangement. If $\bigcap_{H\in\mathcal{B}} H \neq \emptyset$, we call $L := \bigcap_{H\in\mathcal{B}} H$ a (non-empty) intersection of the arrangement $\mathcal{A}$. We define $\bigcap_{H\in\emptyset} H := \mathbb{R}^n$. We denote the set of all this intersections by $\mathcal{E}(\mathcal{A})$.

Sometimes we are not interested in the whole arrangement $\mathcal{A}$, but in some of its subsets:

Definition 2.6. Let $L \in \mathcal{E}(\mathcal{A})$. Then $\mathcal{A}^L := \{H \in \mathcal{A} \mid L \subseteq H\}$ denotes the localisation of $\mathcal{A}$ in L.

The localisation of $\mathcal{A}$ in any intersection $L \in \mathcal{E}(\mathcal{A})$ is by definition a central arrangement.

We define the restriction of an arrangement $\mathcal{A}$ to one of its intersections the following way:

Definition 2.7. Let $L \in \mathcal{E}(\mathcal{A})$. Then $r(\mathcal{A}, L) := \{L \cap H \mid L \nsubseteq H\}$ is called the restriction of $\mathcal{A}$ to L.

$r(\mathcal{A}, L)$ is isomorphic to an arrangement in $\mathbb{R}^{\dim(L)}$.
By restricting an arrangement $\mathcal{A}$ to its intersections, we define the faces of $\mathcal{A}$:

Definition 2.8. Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^n$. A non-empty subset F of $\mathbb{R}^n$ is called a face of $\mathcal{A}$ if F is a region of a restriction of $\mathcal{A}$ to an intersection of $\mathcal{A}$. By convention $\emptyset$ is also called a face of $\mathcal{A}$.

Note that because of this definition faces are considered as relative open sets throughout this thesis. By the definition, the regions of $\mathcal{A}$ are n-dimensional faces of the arrangement $\mathcal{A}$. Therefore, it holds that $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{F}(\mathcal{A})$.

Definition 2.9. Let $F \in \mathcal{F}(\mathcal{A})$ be a face of the arrangement $\mathcal{A}$. Then

$$star(F) := \{F' \in \mathcal{F}(\mathcal{A}) \mid \bar{F}' \cap F = F\} .$$

Definition 2.10. Let $\mathcal{F}$ and $\mathcal{F}'$ be two sets of faces (or regions) of an arrangements $\mathcal{A}$ and $\mathcal{A}'$ respectively. We call $\mathcal{F}$ and $\mathcal{F}'$ combinatorially isomorphic if there is a bijection φ from $\mathcal{F}$ to $\mathcal{F}'$ such that

- codim $(F) =$ codim $(\varphi(F))$ for every $F \in \mathcal{F}$
- $\overline{F} \cap \overline{G} \neq \emptyset \Leftrightarrow \overline{\varphi(F)} \cap \overline{\varphi(G)} \neq \emptyset$ for every $F, G \in \mathcal{F}$.

For every $F \in \mathcal{F}(\mathcal{A})$ exists and intersection $L \in \mathcal{E}(\mathcal{A})$ such that $F \subseteq L$ and $\dim(F) = \dim(L)$. Therefore, $star(F)$ is combinatorially isomorphic to $\mathcal{F}(\mathcal{A}^L)$.

The following definition plays a central role in this thesis:

Definition 2.11. Let $\mathcal{C} \subseteq \mathcal{A}$ denote a central subarrangement of $\mathcal{A}$ and $\mathcal{K} \in \mathcal{P}(\mathcal{C})$ a region of $\mathcal{C}$. Then $\mathcal{K}$ is called a cone of the arrangement $\mathcal{A}$.

Let everything like in Definition 2.11. We make the following restrictions to the sets defined above:

Definition 2.12. Let $\mathcal{A}$ be an arrangement and $\mathcal{K}$ a cone of $\mathcal{A}$.

(i) Then $\mathcal{A}_{\mathcal{K}}$ denotes the set of hyperplanes H of $\mathcal{A}$ which fulfil the condition $H \cap \mathcal{K} \neq \emptyset$:
$$\mathcal{A}_{\mathcal{K}} := \{H \in \mathcal{A} \mid H \cap \mathcal{K} \neq \emptyset\} .$$

(ii) The set of regions of $\mathcal{A}$, which are contained in $\mathcal{K}$, is denoted by $\mathcal{P}_{\mathcal{K}}(\mathcal{A})$:

$$\mathcal{P}_{\mathcal{K}}(\mathcal{A}) := \{P \in \mathcal{P}(\mathcal{A}) \mid P \subseteq \mathcal{K}\} \ .$$

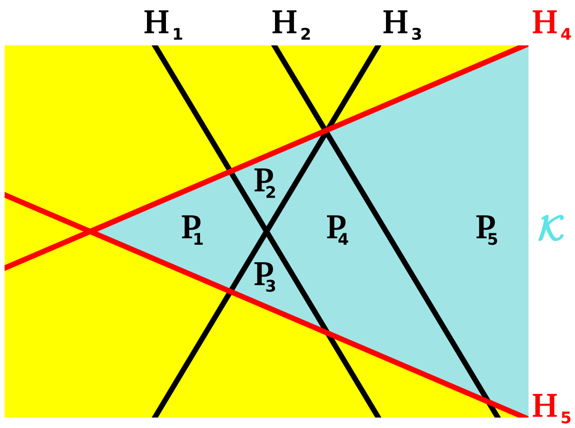
(iii) Let $L \in \mathcal{E}(\mathcal{A})$ such that $L \cap \mathcal{K} \neq \emptyset$ then we call L an intersection of the arrangement $\mathcal{A}$ which crosses the cone $\mathcal{K}$. The set of all $L \in \mathcal{E}(\mathcal{A})$ which cross $\mathcal{K}$ is denoted by $\mathcal{E}_{\mathcal{K}}(\mathcal{A})$:

$$\mathcal{E}_{\mathcal{K}}(\mathcal{A}) := \{L \in \mathcal{E}(\mathcal{A}) \mid L \cap \mathcal{K} \neq \emptyset\} \ .$$

If the subarrangement $\mathcal{C}$ is the empty arrangement, then the only element of $\mathcal{P}_{\mathcal{K}}(\mathcal{C})$ is the whole space $\mathbb{R}^n$. In this case we simplify the notation by writing:

$$\begin{gathered} \mathcal{A} \text{ instead of } \mathcal{A}_{\mathbb{R}^n} \\ \mathcal{P}(\mathcal{A}) \text{ instead of } \mathcal{P}_{\mathbb{R}^n}(\mathcal{A}) \\ \mathcal{E}(\mathcal{A}) \text{ instead of } \mathcal{E}_{\mathbb{R}^n}(\mathcal{A}) \ . \end{gathered}$$

We consider a concrete example for illustrating the definitions from above.



$$\begin{aligned} &\mathcal{A} = \{H_1, H_2, H_3, H_4, H_5\} \\ &\mathcal{C} = \{H_4, H_5\} \\ &\mathcal{A}_{\mathcal{K}} = \{H_1, H_2, H_3\} \\ &\mathcal{P}_{\mathcal{K}}(\mathcal{A}) = \{P_1, P_2, P_3, P_4, P_5\} \\ &\mathcal{E}(\mathcal{A}) = \{\mathbb{R}^2, H_1, H_2, H_3, H_4, H_5, \\ &\qquad H_1 \cap H_3, H_1 \cap H_4, H_1 \cap H_5, \\ &\qquad H_2 \cap H_5, H_3 \cap H_5, H_4 \cap H_5, \\ &\qquad H_2 \cap H_3 \cap H_4\} \\ &\mathcal{E}_{\mathcal{K}}(\mathcal{A}) = \{\mathbb{R}^2, H_1, H_2, H_3, H_1 \cap H_3\} \end{aligned}$$

Figure 2.I: The blue region is the considered cone $\mathcal{K}$. Note that the situation inside $\mathcal{K}$ cannot be created without using a cone.

Example 2.13. Let the arrangement $\mathcal{A}$ and the central subarrangement $\mathcal{C}$ be as in Figure 2.I. By the choice of the central subarrangement $\mathcal{C}$, there are four possible cones of the arrangement $\mathcal{A}$. We consider the light blue one. The three hyperplanes H_1, H_2, H_3 create five regions inside the cone $\mathcal{K}$. Recognize that this can only happen inside a cone, since an arrangement of three hyperplanes has four regions (all hyperplanes are parallel), six regions (two parallel hyperplanes or all hyperplanes intersect in the same point) or seven regions (the hyperplanes are in general position). Therefore, the concept of using cones of an arrangement $\mathcal{A}$ provides a wider range of combinatorial situations than considering only full arrangements of hyperplanes.

We introduce the concept of coning and deconing, which will play a central role in some proofs in Chapter 3 and Chapter 4.

Definition 2.14. Let $\mathcal{A}$ in $\mathbb{R}^n$ be a central arrangement, $H \in \mathcal{A}$ and Z an arbitrary hyperplane parallel to H. Then each hyperplane of $\mathcal{A}$ except H intersects Z non-trivially in a codimension 1 subspace of Z. The arrangement thereby induced in Z is called the decone of $\mathcal{A}$ with respect to Z and is denoted by $\mathcal{A}_Z$.

Because of the construction the decone is isomorphic to an arrangement of hyperplanes in $\mathbb{R}^{n-1}$. Note, that the decone of a central arrangement is not a central arrangement in general.

We describe the inverse geometrical construction, called coning:

Definition 2.15. Let $\mathcal{A}$ in $\mathbb{R}^n$ be an arbitrary arrangement of hyperplanes. Then we construct an arrangement $c\mathcal{A}$ in $\mathbb{R}^{n+1}$ by homogenizing the defining functions by an additional variable x_{n+1} and adding a new hyperplane $\hat{H}$ defined by $f(x) = x_{n+1}$. The arrangement $c\mathcal{A}$ is called the cone over the arrangement $\mathcal{A}$.

The arrangement $c\mathcal{A}$ is a central arrangement, since for every homogenized polynomial $\hat{f}$ it holds that $\hat{f}(0) = 0$. Therefore, 0 is contained in the intersection $\bigcap_{H \in c\mathcal{A}} H$, which means $\bigcap_{H \in c\mathcal{A}} H \neq \emptyset$. The set of faces of the original arrangement $\mathcal{A}$ is combinatorially isomorphic to the situation which is created in an arbitrary halfspace defined by $\hat{H}$. We get back to the original arrangement $\mathcal{A}$ by setting x_{n+1} to 1. This means we are using the construction of deconing by choosing $Z := \{x \in \mathbb{R}^{n+1} \mid x_{n+1} = 1\}$. Thus Z is a parallel to the hyperplane, which is defined by $f(x) = x_{n+1}$.
See Figure 2.II, Figure 2.III and Figure 2.IV for a sketch of the situation.

We introduce some concepts from the theory of oriented matroids. For information on this subject see [4]. Let $\mathcal{A} := \{H_1, \ldots, H_r\}$. We assign to each hyperplane $H_d \in \mathcal{A}$ a positive halfspace, denoted by H_d^+ and a negative halfspace, denoted by H_d^-. Hence we are able to identify each region $P \in \mathcal{P}(\mathcal{A})$ with its sign vector:

$$P \leftrightarrow ((P)_1, \ldots, (P)_r)$$
$$(P)_i = \begin{cases} +, & \text{if } P \subseteq H_i^+ \\ -, & \text{if } P \subseteq H_i^- \end{cases}$$

In other words: The component $(P)_i$ of the sign vector equals the sign of the halfspace of H_i in which the region P is contained.
We extend this definition to the set of faces $\mathcal{F}(\mathcal{A})$. Each face is described uniquely by its sign vector the following way:

$$F \leftrightarrow ((F)_1, \ldots, (F)_r)$$
$$(F)_i = \begin{cases} +, & \text{if } F \subseteq H_i^+ \\ -, & \text{if } F \subseteq H_i^- \\ 0, & \text{if } F \subseteq H_i \end{cases}$$

Figure 2.II: An arrangement consisting of the hyperplanes H_1 and H_2 in $\mathbb{R}^1$. The vector-space $\mathbb{R}^1$ is symbolised by the blue line.

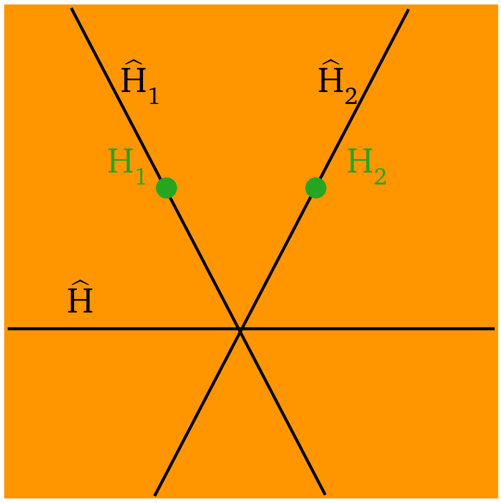


Figure 2.III: By coning the arrangment from Figure 2.II we obtain a central arrangement in $\mathbb{R}^2$ (symbolised by the orange square). The additional hyperplane is denoted by $\hat{H}$ and the hyperplanes which are induced by H_1 and H_2 are denoted by $\hat{H}_1$ and by $\hat{H}_2$.

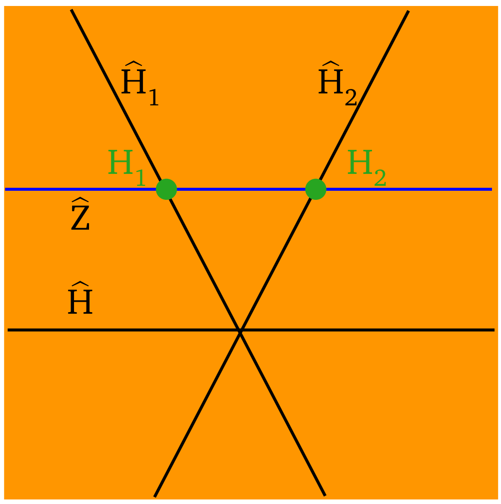


Figure 2.IV: From the arrangement consisting of $\hat{H}$, $\hat{H}_1$ and $\hat{H}_2$ we can go back the arrangement in Figure 2.II by deconing. We intersect the hyperplanes $\hat{H}_1$ and $\hat{H}_2$ by the hyperplane $\hat{Z}$, which is parallel to $\hat{H}$.

Since regions are n-dimensional faces, this definition equals for n-dimensional faces the definition of the sign vector of a region. We define the following multiplication on the set of faces: For $F_1, F_2 \in \mathcal{F}(\mathcal{A})$ we set

$$(F_1 \circ F_2)_i = \begin{cases} (F_1)_i & \text{if } (F_1)_i \neq 0 \\ (F_2)_i & \text{else} \end{cases}$$

This multiplication can be interpreted as a projection: The region F_2 is projected onto a region in $star(F_1)$. We will use this multiplication for showing some results concerning the gate property (see Lemma 3.20).

Example 2.16. We assign to each hyperplane $H \in \mathcal{A}$ a negative and positive halfspace for obtaining the signs vectors of the regions (see Figure 2.V). Note,

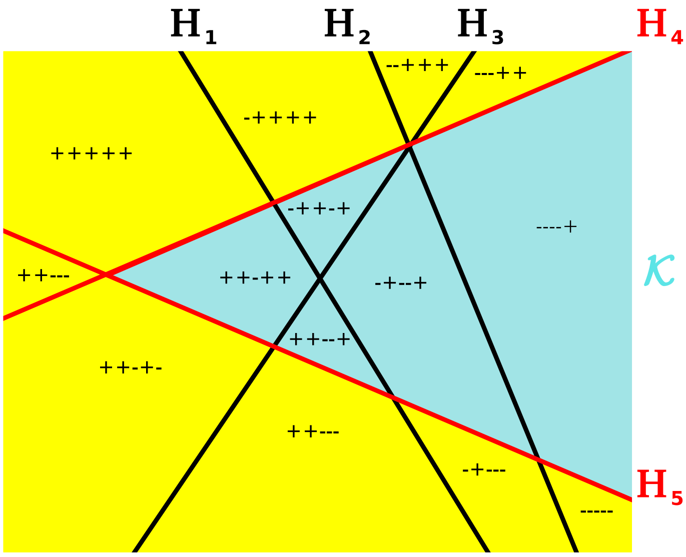


Figure 2.V: The sign vectors are obtained by the choice of a positive and a negative halfspace for each hyperplane.

that the regions inside the cone $\mathcal{K}$ all have the same sign concerning the entries in the sign vector for H_4 and H_5.
Let $F := H_1 \cap H_3$. Then its sign vector is $(0, +, 0, -, +)$. We multiply the region $(-, -, -, -, +)$ by F:

$$(0, +, 0, -, +) \circ (-, -, -, -, +) = (-, +, -, -, +) \ .$$

The result of this multiplication is the region inside $star(F)$, which is the closest one to $(-, -, -, -, +)$ in the sense that the number of hyperplanes which are separating is minimal.

2.2 Polyhedra, polytopes and line shelling

For general information on polyhedra, polytopes and line shelling see [23]. We start by defining polyhedra and polytopes:

Definition 2.17. A polyhedron is an intersection of finitely many open halfspaces in $\mathbb{R}^n$. If the polyhedron is bounded, it is a polytope.

Note, that in this thesis polyhedra and polytopes are open subsets of the $\mathbb{R}^n$.

Definition 2.18. Let P be a polyhedron (resp. polytope) and H be a hyperplane, such that P is entirely contained in one of the two half-spaces defined by H. Then the face defined by H is the relative interior of $\bar{P} \cap H$.
Faces of dimension one less than the dimension of the polyhedron are called facets. The polyhedron itself is a face by definition.

By this definition we see, that the faces of a polyhedron are polyhedra again.

Definition 2.19. A polyhedral complex C is a finite collection of polyhedra in $\mathbb{R}^n$, such that

(i) the empty polyhedron is in C,

(ii) if $P \in \mathsf{C}$, then all the faces of P are also in C,

(iii) the intersection $\bar{P} \cap \bar{Q}$ of the closure of two polyhedra $P, Q \in \mathsf{C}$ is a face of P and Q.

The underlying set of C is the union its faces:

$$||\mathsf{C}|| := \bigcup_{F \in \mathsf{C}} F \ .$$

The dimension $\dim \mathsf{C}$ is the largest dimension of a polyhedron in C.

A so called polytopal complex is the special case of a polyhedral complex:

Definition 2.20. Let C be a polyhedral complex in $\mathbb{R}^n$. If all the polyhedra in C are bounded - and therefore polytopes - C is a polytopal complex.

Concerning polytopes, we are especially interested in the following two polytopal complexes:

Definition 2.21. Let P be a polytope. By $\mathsf{C}(\bar{P})$ we denote the polytopal complex consisting of all faces of P. The boundary complex of P is denoted by $\mathsf{C}(\partial\bar{P})$ and it consists of all the faces of P besides the polytope itself.

We can store some information about a polyhedral complex in its f-vector:

Definition 2.22. Let C be a polyhedral complex of dimension n. Then

$$(f_{-1}, f_0, \dots, f_n)$$

where f_i denotes the number of faces of dimension i for $-1 \leq i \leq n$. By the f-vector of a polytope P we mean the f-vector of its boundary complex $\mathsf{C}(\partial\bar{P})$.

We introduce some helpful results by Bruggeser and Mani. For the original paper see [6] and for secondary literature [23]. First, we define what we understand by a shelling. We restrict our self to defining shellability for the boundary complex of a polytop. For a more general definition see [23].

Definition 2.23. Let P be a polytope in $\mathbb{R}^n$ with $\dim(P) = n$ and $\mathsf{C}(\partial \bar{P})$ the associated boundary complex. A shelling of $\mathsf{C}(\partial \bar{P})$ is a linear ordering $F_1, F_2, \ldots, F_s$ of the facets of P such that either P is $0-$dimensional, or it satisfies the following conditions:

- The boundary complex $\mathsf{C}(\partial \bar{F}_1)$ of the first facet F_1 has a shelling.
- For $1 < j \leq s$ the intersection of the closure $\bar{F}_j$ of the facet F_j with the union of the closure of the previous facets is a non-empty polytopal complex whose facets are the beginning segment of a shelling of the $(k-1)$-dimensional boundary complex of $\bar{F}_j$, that is

$$\bar{F}_j \cap \left(\bigcup_{i=1}^{j-1} \bar{F}_i \right) = \bar{G}_1 \cup \bar{G}_2 \cup \ldots \cup \bar{G}_r$$

  for some shelling $G_1, G_2, \ldots, G_r, \ldots, G_t$ of $\mathsf{C}(\partial \bar{F}_j)$, and $1 \leq r \leq t$.

P is shellable, if its boundary complex $\mathsf{C}(\partial \bar{P})$ has a shelling.

Concerning polyoptes Bruggeser and Mani provide the following helpful result:

Theorem 2.24 (Bruggeser and Mani)**.** *Polytopes are shellable.*

A shelling of a polytope can be found by a geometrical construction, called line shelling.
First, we define the visibility of a facet F of a polytope P with respect to a point in general position:

Definition 2.25. Let P be a polytope in $\mathbb{R}^n$, F a facet of P and $x \in \mathbb{R}^n$ a point outside $\bar{P}$, such that it lies in general position (that is, not in the affine hull of a facet of P). Then F is visible from x if and only if x and P are on different sides of the hyperplane spanned by F.

We recall the corresponding theorem from [23] and then we explain how to construct a line shelling.

Theorem 2.26. *Let $P \subseteq \mathbb{R}^n$ be a n-polytope, and let $x \in \mathbb{R}^n$ be a point outside P in general position. Then the boundary complex $\mathsf{C}(\partial \bar{P})$ has a shelling in which the facets of P that are visible from x come first.*

Ziegler provides in [23] many geometric sketches for explaining this construction. Here, we give a brief description of the geometrical construction for a line shelling.

Let P be a polytope and x a point outside P in general position and p a point inside P, such that we are able to draw a line l from x through p with the property that its intersection points with the hyperplanes, which are spanned by the facets of P are distinct. We demand in addition that l is not parallel to any of this hyperplanes, for avoiding an intersection point at infinity. Let F, F' be facets of P and H, H' the hyperplanes spanned by F and F'. We define a partial ordering on the facets by:

$$F \leq F' \Leftrightarrow [p, l \cap H] \subset [p, l \cap H']$$

(see also Figure 2.VI). Imagine, we are walking along the line l from p to x. Then the ordering is exactly the order of the faces, in which they become visible while we are walking along l. We walk along the line until we pass through infinity and come back to the polytope from the opposite side. Therefore, we have to extend the line from p in the opposite direction concerning p. Again, let F, F' be facets of P and H, H' the hyperplanes spanned by F and F'. We define a partial ordering on the facets by:

$$F \leq F' \Leftrightarrow [p, l \cap H] \supset [p, l \cap H'] \ .$$

This means, we order the facets by the order of their disappearing (see also Figure 2.VII). The combination of the two partial orders gives a total order of the facets of P. Therefore, by this geometrical construction we get order of the facets of P, which is a shelling (see [23] for further details).

The following theorem gives a nice fact concerning shelling orders of a polytope:

Lemma 2.27. *If $F_1, F_2, \ldots, F_s$ is a shelling order for the boundary complex of a polytope P, then so is the reverse order $F_s, F_{s-1}, \ldots, F_1$.*

We define the reduced Euler characteristic of a polyhedral complex:

Definition 2.28. Let C be a polyhedral complex of dimension n with f-vector $(f_{-1}, f_1, \ldots, f_n)$. Then the reduced Euler characteristic of C is defined by:

$$\tilde{\chi}(\mathsf{C}) := -f_{-1} + f_0 - f_1 + \ldots + (-1)^n f_n \ .$$

The value of this Euler characteristic is the same as the value of the topological Euler characteristic of the underlying space $||\mathsf{C}||$, see [11], Thereom 22.2 .

In the case of a polytope, we use the existence of a line shelling for computing the Euler characteristic of the polytopal complex $\mathsf{C}(\partial \bar{P})$ (see also [23]).

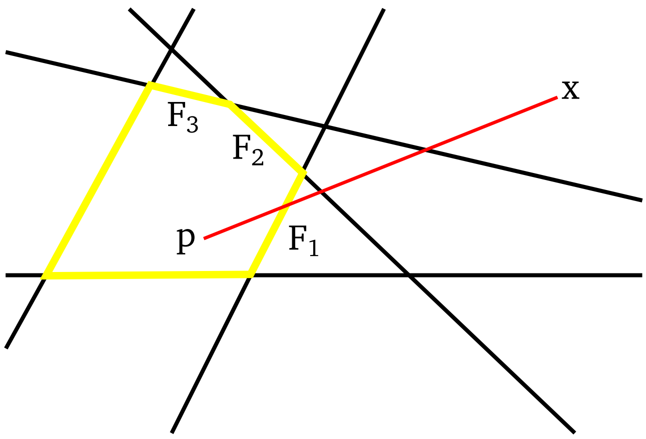


Figure 2.VI: Consider the yellow marked boundary of a polytope. The red line from p to x defines an ordering on the facets which are visible from x.

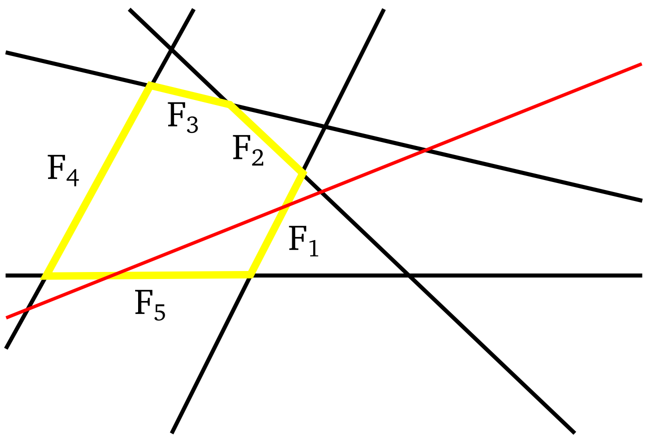


Figure 2.VII: We extend the red line and come back from the opposite direction. We obtain an ordering of the facets, which were not visible in the previous step of the construction.

Lemma 2.29. *Let P be a polytope of dimension n with f-vector $(f_{-1}, f_0, \dots, f_{n-1})$. Let $F_1, \dots, F_{n-1}$ be a shelling of the polytopal complex $\mathsf{C}(\partial \bar{P})$. Then it holds that*

$$\tilde{\chi}(\mathsf{C}(\bar{F}_1 \cup \ldots \cup \bar{F}_j)) = \begin{cases} 0, & \text{if } 1 \leq j < f_{n-1} \\ (-1)^{n-1}, & \text{if } j = f_{n-1} \ . \end{cases}$$

For the proof of this lemma, we need the following one:

Lemma 2.30. *Let P be a polytope of dimension n with f-vector $(f_{-1}, f_0, \dots, f_{n-1})$. Let $F_1, \dots, F_{f_{n-1}}$ be a shelling of the polytopal complex $\mathsf{C}(\partial \bar{P})$. Then for $j < f_{n-1}$ it holds:*

$$\bar{F}_j \cap \left(\bigcup_{i<j} \bar{F}_i \right)$$

is a shellable part of, but not the whole boundary of F_j.

Proof. Let $F_1, \dots, F_{f_{n-1}}$ be a shelling of the polytopal complex $\mathsf{C}(\partial \bar{P})$. Then by the definition of a shelling, it holds that $\bar{F}_j \cap \left(\bigcup_{i<j} \bar{F}_i \right)$ is shellable for $1 \leq j \leq f_{n-1}$. Now, let $j < f_{n-1}$. Assume, that $\bar{F}_j \cap \left(\bigcup_{i<j} \bar{F}_i \right)$ is the whole boundary complex of $\bar{F}_j$. This means, that all facets F_i such that $\bar{F}_i \cap \bar{F}_j$ is a facet of F_j come earlier than F_j in the shelling order. By Lemma 2.27 the reverse ordering $F_{f_{n-1}}, \dots, F_1$ is a shelling, too. In this shelling the facets F_i such that $\bar{F}_i \cap \bar{F}_j$ is a facet of F_j come earlier later F_j. Therefore, $\bar{F}_j \cap \left(\bigcup_{j<i} \bar{F}_i \right)$ is not a union of the closure of facets of F_j. This contradicts the definition of a shelling. Thus, the lemma is proved. ☐

Now we are in position to prove Lemma 2.29.

Proof. We show the lemma by induction on the dimension of the polytope P. Let $\dim P = 1$ then $\dim \mathsf{C}(\partial \bar{P}) = 0$ and and $\mathsf{C}(\partial \bar{P})$ is a 1-sphere. Therefore, it consists of two disjoint 0-dimensional faces F_1, F_2. This faces are 0-dimensional polytopes. Hence, they are 0-disks. Since the Euler characteristic of a 1-sphere and a 0-disk are well-known, we get

$$\tilde{\chi}(\mathsf{C}(\partial \bar{P})) = 1$$
$$\tilde{\chi}(\mathsf{C}(\bar{F}_i)) = 0 \text{ for } i \in \{1, 2\} \ .$$

Now let P be an n-dimensional polytope with shelling order $F_1, \dots, F_{f_{n-1}}$. We do a second induction on the linear order of the facets. Since F_j is an $(n-1)$-sphere for $1 \leq j \leq f_{n-1}$, it holds that

$$\tilde{\chi}(\mathsf{C}\bar{F}_1) = 0 \ .$$

Consider

$$\tilde{\chi}(\mathsf{C}(\bar{F}_1 \cup \ldots \cup \bar{F}_j))$$

By the additivity of the reduced Euler characteristic we get

$$\tilde{\chi}(\mathsf{C}(\bar{F}_1 \cup \ldots \cup \bar{F}_j)) = \tilde{\chi}(\mathsf{C}(\bar{F}_1 \cup \ldots \cup \overline{F_{j-1}})) + \tilde{\chi}(\mathsf{C}\bar{F}_j) - \tilde{\chi}(\mathsf{C}(\bar{F}_j \cap (\bar{F}_1 \cup \ldots \cup \overline{F_{j-1}}))$$

The first summand equals 0 by induction hypothesis of the second induction. The second summand equals 0, because F_j is an $(n-1)$-sphere. By Lemma 2.30 is the polytopal complex $\mathsf{C}(\bar{F}_j \cap (\bar{F}_1 \cup \ldots \cup \overline{F_{j-1}}))$ only a part, but not the whole boundary of $\bar{F}_j$ for $j < f_{n-1}$. Therefore, by the induction hypothesis of the exterior induction, it holds that

$$\tilde{\chi}(\mathsf{C}(\bar{F}_j \cap (\bar{F}_1 \cup \ldots \cup \overline{F_{j-1}}))) = \begin{cases} 0, & \text{if } j < f_{n-1} \\ (-1)^{n-2}, & \text{if } j = f_{n-1} \end{cases} .$$

Altogether, we get

$$\tilde{\chi}(\mathsf{C}(\bar{F}_1 \cup \ldots \cup \bar{F}_j)) = \begin{cases} 0, & \text{if } j < f_{n-1} \\ (-1)^{n-1}, & \text{if } j = f_{n-1} \end{cases} .$$

Therefore, Lemma 2.29 is proved. $\square$

2.3 Partially ordered sets, simplicial complexes and topology

Partially ordered sets are used in Chapter 3 as a tool for some proofs and in Chapter 5 we apply Theorem 4.5 to a situation which is created by certain posets. Here, we start with the basic definitions:

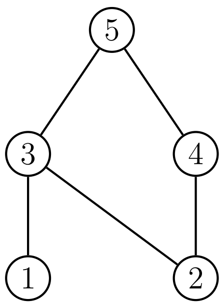
Definition 2.31. Let X be a set. A relation $\leq$ on $X \times X$ is called a partial order, if the following three conditions hold:

- $x \leq x$ for every element x of X.
  (reflexive)
- if $x \leq y$ and $y \leq x$, then $x = y$ for every two elements x and y of X.
  (anti-symmetric)
- if $x \leq y$ and $y \leq z$, then $x \leq z$ for every three elements x, y and z of X.
  (transitive)

Definition 2.32. We say $P := (X, \leq)$ is a partially ordered set or poset for short if X is a (finite) set that is partially ordered by an order relation $\leq$.

Definition 2.33. Let $P = (X, \leq)$ be a poset and $x, y \in X$ such that $x \neq y$. We say $x \leq y$ is a covering, if there is no $z \in X$ such that $x \neq z$, $y \neq z$ and $x \leq z \leq y$.

In the following we will always consider finite posets. A finite poset $P = (X, \leq)$ is defined by its coverings, because the ordering $\leq$ is transitive. We use this fact for visualizing posets in the so-called Hasse diagrams:



$X := \{1, 2, 3, 4, 5\}$
non-trivial ordering relations:
$1 \leq 3$, $1 \leq 5$, $2 \leq 3$, $2 \leq 4$, $2 \leq 5$, $3 \leq 5$, $4 \leq 5$
coverings:
$1 \leq 3$, $2 \leq 3$, $2 \leq 4$, $3 \leq 5$
incomparable:
$1||2$, $1||4$, $3||4$
antichains:
$A_1 := \{1, 2\}$, $A_2 := \{1, 4\}$, $A_3 := \{3, 4\}$
linear extensions:
$1 \leq 2 \leq 3 \leq 4 \leq 5$, $2 \leq 1 \leq 3 \leq 4 \leq 5$, $1 \leq 2 \leq 4 \leq 3 \leq 5$,
$2 \leq 1 \leq 4 \leq 3 \leq 5$, $2 \leq 4 \leq 1 \leq 3 \leq 5$

Figure 2.VIII: Example for the Hasse Diagram of a partially ordered set and for some basic definitions.

Usually in a partially ordered set, there exist some elements, which are incomparable:

Definition 2.34. Let $P := (X, \leq)$ be a partially ordered set and $x, y \in X$ such that neither $x \leq y$ nor $y \leq x$. Then x and y are incomparable. We write $x||y$.

Definition 2.35. Let $P := (X, \leq)$ be a partially ordered set and $A \subseteq X$ such that $x||y$ for every pair $x, y \in A$ with $x \neq y$. Then A is an antichain.

A special type of posets will be of particular interest in Chapter 5:

Definition 2.36. A linear extension of a poset $P := (X, \leq)$ is a poset $(X, \leq^\star)$, such that

(i) $x \leq y \Rightarrow x \leq^\star y$

(ii) for $x, y \in X$ holds $x \leq^* y$ or $y \leq^* x$.

We recall some of the basics concerning simplicial complexes. We will need them later on for some topological connections.

Definition 2.37. An abstract simplicial complex Δ on a finite vertex set V is a non-empty collection of subsets of V such that

- $\{v\} \in \Delta$ for all $v \in V$
- if $G \in \Delta$ and $F \subseteq G$ then $F \in \Delta$.

The elements of Δ are called faces and the maximal faces are called facets.

Definition 2.38. Let F be a face of a simplicial complex Δ. The dimension of F is defined by

$$\dim F = |F| - 1 \ .$$

The dimension of the simplicial complex Δ is defined by

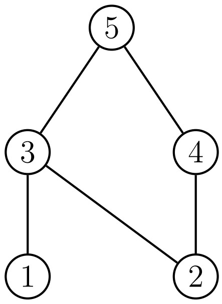
$$\dim \Delta = \max_{F \in \Delta} \dim F \ .$$

Definition 2.39. Let P be a poset. Then the totally ordered subsets of P are called the chains of P.

The following definition shows a connection between posets and simplicial complexes:

Definition 2.40. One can associate an abstract simplicial complex $\Delta(P)$ to every poset P: The vertices of $\Delta(P)$ are the elements of P and the faces are the chains of P. We call this simplicial complex the order complex of the poset P.

In Chapter 3 we need some topological properties of posets. We recall the used and well-known theorems. Since we are not particularly interested in the groundset of the poset, we write P instead of $P := (X, \leq)$ in the following lemmata. The notation $x \in P$ is a short version of $P := (X, \leq)$ and $x \in X$.



$1 \leq 3,\ 1 \leq 5,\ 2 \leq 3,$
$3 \leq 5,\ 2 \leq 4,\ 2 \leq 5,$
$4 \leq 5,$
$1 \leq 3 \leq 5,$
$2 \leq 3 \leq 5,$
$2 \leq 4 \leq 5$

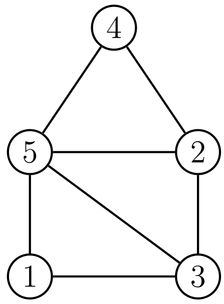


Figure 2.IX: A poset (on the left), its non-trivial chains (in the middle) and its order complex (on the right).

In order to link an abstract simplicial complex Δ to topology we associate to Δ a polytopal complex. We consider $\mathbb{R}^V$ with basis vectors e_v, $v \in V$ and to $F \in \Delta$ we associate the geometric simplicial complex spanned by e_v, $v \in F$. The collection of the relative interiors of all these geometric simplices is a polytopal complex $\mathsf{C}(\Delta)$ and the set theoretic of its elements with the subspace topology is called the geometric realisation $||\mathsf{C}(\Delta)||$ of Δ. Usually we abbreviate this by $||\Delta||$. For this connections see [11].

We start with the definition of the reduced Euler characteristic of a simplicial complex, which is a topological invariant.

Definition 2.41. Let Δ be a simplicial complex with f-vector

$$(f_{-1}, f_0, \ldots, f_{\dim(\Delta)}) \ .$$

Then the reduced Euler characteristic of Δ is defined by

$$\tilde{\chi}(\Delta) = \sum_{i=-1}^{\dim(\Delta)} (-1)^i \cdot f_i \ .$$

Of course $\tilde{\chi}(\Delta) = \tilde{\chi}(\mathsf{C}(\Delta))$ as defined in Definition 2.28.

If we are interested in the reduced Euler characteristic of the order complex of a poset, the following lemma is helpful, since it allows us to switch over under certain circumstances to another poset for doing the calculation.

Lemma 2.42 (Quillen's Fibre Lemma, [14, 21])**.** *Let P, Q be posets and $f : P \to Q$ a poset map, i.e. an order-preserving map. If the fibre $f^{-1}(Q_{\leq y})$ is contractible for all $y \in Q$ then*

$$||\Delta P|| \simeq ||\Delta Q|| \ .$$

In other words, $||\Delta P||$ and $||\Delta Q||$ are homotopy equivalent and therefore they have the same Euler characteristic.

We define an invariant of a poset P:

Definition 2.43. Let P be a poset and $x, y \in P$ such that $x \leq y$. Then the Möbius function $\mu_P(x, y)$ is defined as follows:

$$\mu_P(x,y) = \begin{cases} 1, & \text{if } x = y \\ -\sum_{x \leq z < y} \mu_P(x,z), & \text{else} \end{cases}$$

There is a connection between the Möbius function of a poset P and the reduced Euler characteristic of its order complex ΔP. Before we recall the lemma, we have to define the bounded extension of a poset.

Definition 2.44. Let P be a poset. Then $\hat{P}$ denotes the bounded extension of P:

$$\hat{P} = P \cup \{\hat{0}, \hat{1}\} \ .$$

The following shows the announced connection:

Lemma 2.45 (Philip Hall Theorem [21, 3])**.** *For any poset P*

$$\mu_{\hat{P}}(\hat{0}, \hat{1}) = \tilde{\chi}(\Delta(P)) \ .$$

We can make each polyhedral complex C into a poset $P(\mathsf{C})$ by using the ordering

$$x \leq y \Leftrightarrow \bar{x} \subseteq \bar{y}$$

for $x, y \in \mathsf{C}$. Since every polyhedral complex is a regular CW-complex, the following lemma gives us an important connection between C and $P(\mathsf{C})$. For the definition of a CW-complex see [11], Chapter 38.

Lemma 2.46 (see [3])**.** *Let K be a regular CW-complex and $P(K)$ the poset obtained by ordering the closed cells of K by inclusion. Then*

$$||\Delta(P(K))|| \cong K \ .$$

In other words, $||\Delta(P(K))||$ and K are homeomorphic.

Since $||\Delta(P(K))||$ and K are homeomorphic, they are especially homotopy equivalent and therefore they have the same Euler characteristic.

In the end of this chapter, we give two lemmata about the Möbius function of a poset.

Definition 2.47. Let P be a poset. A map $^{-}\colon P \to P$ with $x \mapsto \bar{x}$ is called a closure operation (or short "closure") on P, if it fulfils the following conditions

(i) $x \leq \bar{x}$ for every element x of P

(ii) $x \leq y \Rightarrow \bar{x} \leq \bar{y}$ for every two elements x, y of P

(iii) $\bar{\bar{x}} = \bar{x}$ for every element x of P.

Definition 2.48. Let P be a poset and $^-\colon P \to P$ with $x \mapsto \bar{x}$ a closure operation on P. The set

$$\{x \in P \mid \bar{x} = x\}$$

is called the quotient of P.

We recall the well-known theorem from Rota ([2, p.167]) about closure operations:

Lemma 2.49 (Rota)**.** *Let P be a locally finite poset and $x \mapsto \bar{x}$ a closure (operation) on P with quotient Q. Then for all $x, y \in P$:*

$$\sum_{z \in P, \bar{z} = \bar{y}} \mu(x, z) = \begin{cases} \mu_Q(\bar{x}, \bar{y}), & \text{if } x = \bar{x} \\ 0, & \text{if } x < \bar{x} \end{cases}$$

Lemma 2.50. *Let $P := (X, \leq)$ be a poset and $x, y \in P$ such that $x < y$. If a $v \in P$ exists such that $x < v < y$ and $\mu(x, v) = 0$ then*

$$\mu_P(x, y) = \mu_{P \setminus \{v\}}(x, y)$$

Although Lemma 2.50 recalls a well-known fact, we give a proof for the sake of completeness:

Proof. We prove Lemma 2.50 by induction. Let $x, v, w \in P$ such that $x < v < w$,$v < w$ is a covering and $\mu_P(x, v) = 0$. Then the following holds:

$$\begin{aligned} \mu_P(x, w) &= -\sum_{x \leq z < w} \mu_P(x, z) \\ &= -\sum_{x \leq z < w, z \neq v} \mu_P(x, z) - \mu_P(x, v) \\ &= \mu_{P \setminus \{v\}}(x, w) \end{aligned}$$

The last equality holds, since we do not use any other element bigger v besides w, because w is the upper bound and covers v. Hence in the recursion $\mu_P(x, v)$ is the only summand depending on v. Now let $w > v$ no covering and $\mu_P(x, z) = \mu_{P \setminus \{v\}}(x, z)$ for $z < w$ by induction hypothesis. Then:

$$\begin{aligned} \mu_P(x, w) &= -\sum_{x \leq z < w} \mu_P(x, z) \\ &= -\sum_{x \leq z < w} \mu_{P \setminus \{v\}}(x, z) \\ &= \mu_{P \setminus \{v\}}(x, w) \end{aligned}$$

$\square$

3 Varchenko's Theorem for central arrangements

The aim of this chapter is to give a proof of Varchenko's Theorem for central arrangements. First, we introduce this theorem (see Theorem 3.5). Then we define Edelman's poset and prove some properties concerning its Möbius function. We need these properties for describing a factorisation of the Varchenko Matrix $B(\mathcal{A})$ in the case of a central arrangement. We give a proof of this factorisation and use it finally for proving the main theorem of this chapter, Theorem 3.5.

For some parts of the proofs in this chapter, we follow some ideas drafted in [7] and amend the argumentation when necessary. For lemmata and theorems which can be found in [7] we will give detailed proofs.

Through the whole chapter $\mathcal{A}$ always denotes a central arrangement as long as it is not explicitly said that $\mathcal{A}$ is an arbitrary arrangement of hyperplanes.

3.1 Varchenko's Theorem for central arrangements

We start with some definitions which we need for formulating Theorem 3.5, the main theorem of this section.
Although we are mainly interested in central arrangements in this sections, the following can be defined for arbitrary arrangements of hyperplanes:

We start with the central definition of the bilinear-form defined by Varchenko in [19]. Let $\mathcal{A}$ be an arrangement of hyperplanes. To every hyperplane $H \in \mathcal{A}$ we assign a weight a_H which we treat as an indeterminate in the ring of polynomials $R_{\mathcal{A}} := \mathbb{Z}[a_H, H \in \mathcal{A}]$. The weight of an intersection $L \in \mathcal{E}(\mathcal{A})$ is denoted by $a(L)$ and defined the following way:

$$a(L) := \prod_{H:L\subseteq H} a_H$$

Let M be the $R_{\mathcal{A}}$-module, which is freely generated by the regions of $\mathcal{P}(\mathcal{A})$.

Definition 3.1. Let $\mathcal{A}$ be an arrangement, $P, Q \in \mathcal{P}(\mathcal{A})$. Then we can define a symmetric $R_{\mathcal{A}}$-bilinear form $B(\cdot, \cdot)$ over the module M by the following:

$$B(P, Q) := \prod_{H \in T(P,Q)} a_H \ ,$$

with $T(P, Q) := \{H \in \mathcal{A} \mid H \text{ seperates } P \text{ and } Q\}$.

We number the regions of $\mathcal{P}(\mathcal{A})$ by $P_1, .., P_r$ and write $B(\mathcal{A})$ for the $r \times r$ matrix with entries

$$B(\mathcal{A})_{i,j} := B(P_i, P_j) \ .$$

We call this matrix the Varchenko matrix.

Definition 3.2. Let $L \in \mathcal{E}(\mathcal{A})$ and $H \in \mathcal{A}$ such that $L \subseteq H$. Let $P \in \mathcal{P}(\mathcal{A})$. We say that L is the minimal intersection containing $\bar{P} \cap H$ if $\bar{P} \cap H \subseteq L$ and $\dim(L) = \dim \bar{P} \cap H$.

Remark 3.3. Let $L \in \mathcal{E}(\mathcal{A})$, $H \in \mathcal{A}$ with $H \supseteq L$ and $P \in \mathcal{P}(\mathcal{A})$ such that $H \cap \bar{P} \subset L$. Then there is always an intersection $L' \subseteq L$ such that $\dim(L') = \dim H \cap \bar{P}$: We assume $\dim H \cap \bar{P} < \dim(L)$. Therefore, $\hat{H} \cap \bar{P} := \bar{F}$ is the closure of a face F of P for which $\dim F < \dim L$ holds. Then there is an intersection $L' := \bigcap_{H \in \mathcal{A}: F \supseteq H} \in \mathcal{E}(\mathcal{A})$ with the property $H \cap \bar{P} \subseteq L'$ and $\dim H \cap \bar{P} = \dim L'$. In other words, for each $P \in \mathcal{P}(\mathcal{A})$ exists an intersection $L \in \mathcal{E}(\mathcal{A})$ such that for $H \in \mathcal{A}$ with $L \subseteq H$ it holds that L is the minimal intersection containing $\bar{P} \cap H$.

To each intersection $L \in \mathcal{E}(\mathcal{A})$ we assign a natural number the following way:

Definition 3.4 (see [7])**.** Let $\mathcal{A}$ be an arrangement and $L \in \mathcal{E}(\mathcal{A})$ a non-empty intersection. Choose a hyperplane which contains L. Then $2 \cdot l_{\mathcal{P}(\mathcal{A})}(L)$ equals the number of regions $P \in \mathcal{P}(\mathcal{A})$, which have the property that L is the minimal (with respect to inclusion) intersection containing $\bar{P} \cap H$.

Let everything be defined like in Definition 3.4 and let $P \in \mathcal{P}(\mathcal{A})$ be a region such that L is the minimal intersection containing $\bar{P} \cap H$. Then $\bar{P} \cap H$ is the closure of a face F of P. Let $(P)_{(-F)}$ denote the image of P under the reflection in the face F. Then $(P)_{(-F)}$ also has the property that L is the minimal intersection containing $\overline{(P)_{(-F)}} \cap H$. Therefore, $2 \cdot l_{\mathcal{P}(\mathcal{A})}(L)$ is even and $l_{\mathcal{P}(\mathcal{A})}(L) \in \mathbb{N}$.

Now we are able to formulate Varchenko's Theorem [19] for central arrangements:

Theorem 3.5. *Let $\mathcal{A}$ be a central arrangement of hyperplanes. Then for the determinant of the bilinear form $B(\cdot, \cdot)$ the following holds:*

$$\det B(\mathcal{A}) = \prod_{L \in \mathcal{E}(\mathcal{A})} (1 - a(L)^2)^{l_{\mathcal{P}(\mathcal{A})}(L)} .$$

Remark 3.6. Let $L := \mathbb{R}^n$. Since there is no hyperplane $H \in \mathcal{A}$ such that $\mathbb{R}^n \subseteq H$, we get

$$l_{\mathcal{P}(\mathcal{A})}(\mathbb{R}^n) = 0 .$$

By the same argumentation the product $\prod_{H: \mathbb{R}^n \subseteq H} a_H$ is empty and therefore

$$a(\mathbb{R}^n) = 1 .$$

Thus

$$(1 - a(\mathbb{R}^n)^2)^{l_{\mathcal{P}(\mathcal{A})}(\mathbb{R}^n)} = (1 - 1)^0 = 1 .$$

Therefore, $\mathbb{R}^n \in \mathcal{E}(\mathcal{A})$ does not contribute to the product in Theorem 3.5. But since usually $\mathbb{R}^n \in \mathcal{E}(\mathcal{A})$ and for keeping the notation simple we will not exclude $\mathbb{R}^n$.

3.2 Edelman's Poset and Möbius Function

In this subsection we define Edelman's poset and prove some properties of its Möbius function which will be needed later on.

Definition 3.7 (Edelman's poset, [9])**.** Let $\mathcal{A}$ be an arrangement of hyperplanes and $\mathcal{P}(\mathcal{A})$ the set of its regions. Like above $T(P,Q)$ is the set containing the hyperplanes which seperate P and Q for $P,Q \in \mathcal{P}(\mathcal{A})$. Then we can equip $\mathcal{P}(\mathcal{A})$ with a poset structure

$$P \leq_R Q \Leftrightarrow T(R,P) \subseteq T(R,Q)$$

We denote this partially ordered set by $\mathcal{EP}(\mathcal{P}(\mathcal{A}), R)$.

Recall the following: Let $\mathcal{A} := \{H_1, \ldots, H_r\}$. Each hyperplane $H_d \in \mathcal{A}$ defines a positive and negative half-space, which are denoted by H_d^+ and H_d^-. Then for each region R we define a vector $(R) = ((R)_1, \ldots, (R)_r) \in \{+,-\}$ by

$$(R)_d := \begin{cases} +, \text{ if } R \subseteq H_d^+ \\ -, \text{ if } R \subseteq H_d^- \end{cases} .$$

As in [7] we define the following restriction of $P(\mathcal{P}(\mathcal{A}), R)$. Fix a hyperplane $H_d \in \mathcal{A}$. Then

$$\mathcal{P}_{-(R)_d}(\mathcal{A}) = \{P \in \mathcal{P}(\mathcal{A}) \mid (P)_d = -(R)_d\} .$$

We restrict Edelman's poset to $\mathcal{P}_{-(R)_d}(\mathcal{A})$ and add a unique minimal element $\hat{0}$. We denote the restricted poset by

$$\mathcal{EP}(\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\}, R) .$$

Remark 3.8. We can identify $\hat{0}$ with the region R.

We define a hyperplane parallel to the hyperplane H_d the following way:

$$\begin{aligned} Z_d^+ &\text{ parallel of } H_d \text{ contained in } H_d^+ \\ Z_d^- &\text{ parallel of } H_d \text{ contained in } H_d^- \end{aligned}$$

All the hyperplanes of $\mathcal{A}$ except H_d intersect $Z_d^{-(R)_d}$ non-trivially and thereby induce an arrangement of hyperplanes in $Z_d^{-(R)_d}$. This is called the decone of $\mathcal{A}$ with respect to H_d and denoted by $\mathcal{A}_{Z_d^{-(R)_d}}$. The decone is independent on the choice of $Z_d^{-(R)_d}$.

We can identify each region in $\mathcal{P}_{-(R)_d}(\mathcal{A})$ with a unique region in $\mathcal{P}(\mathcal{A}_{Z_d^{-(R)_d}})$ via inclusion:

Lemma 3.9. *There exists a bijection*

$$h: \begin{array}{ccc} \mathcal{P}_{-(R)_d}(\mathcal{A}) & \rightarrow & \mathcal{P}(\mathcal{A}_{Z_d^{-(R)_d}}) \\ P & \mapsto & Q \text{ , such that } Q \subset P \end{array}$$

Because of the definition of the decone it follows directly that h is a bijection.

We define on $\mathcal{P}(\mathcal{A}_{Z_d^{-(R)_d}}) \cup \{\hat{0}\}$ a poset structure by the ordering inherited from $\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\}$ and denote this poset by

$$\mathcal{EP}(\mathcal{P}(\mathcal{A}_{Z_d^{-(R)_d}}) \cup \{\hat{0}\}, R) \ .$$

Remark 3.10. The map h defined in Lemma 3.9 is a poset isomorphism between $\mathcal{EP}(\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\}, R)$ and $\mathcal{EP}(\mathcal{P}(\mathcal{A}_{Z_d^{-(R)_d}}) \cup \{\hat{0}\}, R)$.

We are interested in calculating the Möbius function of the poset $\mathcal{EP}(\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\}, R)$. First, we make some simplification concerning the notation.
In the notation of Chapter 2.3 we index the Möbius function with the poset:

$$\mu_{\mathcal{EP}(\mathcal{P}_{-(R)_d}(\mathcal{A})\cup\{\hat{0}\},R)}(\cdot,\cdot)$$

respectively

$$\mu_{\mathcal{EP}(\mathcal{P}(\mathcal{A}_{Z_d^{-(R)_d}})\cup\{\hat{0}\},R)}(\cdot,\cdot) \ .$$

Since the two posets are isomorphic and for simplifying the notation for indexing we will use the bottom element R of Edelman's poset in both cases and write:

$$\mu_R(\cdot,\cdot) \ .$$

We define the following poset, which helps us to determine the Möbius function of the reduced Edelman Poset (see Theorem 3.17).

Definition 3.11 (see also [7, p.171])**.** Let $H_d \in \mathcal{A}$, let $R \in \mathcal{P}(\mathcal{A})$, $P \in \mathcal{P}_{-(R)_d}(\mathcal{A})$ and $h(P), h(-R) \in \mathcal{P}(\mathcal{A}_{Z_d^{-(R)_d}})$. Define $\mathcal{G} := T(h(P), h(-R))$ and $\mathcal{B} := \mathcal{A}_{Z_d^{-(R)_d}} \setminus \mathcal{G}$. Let $B \in \mathcal{P}(\mathcal{B})$ be the unique region, which contains $h(P)$ and $h(-R)$ and let $G \in \mathcal{P}(\mathcal{G})$ the unique region which contains $h(P)$. Then we define the poset

$$\begin{aligned} &W_R(P) = \{F \in \mathcal{F}(\mathcal{A}_{Z_d^{-(R)_d}}) \mid F \subseteq \bar{G} \cap \partial\bar{B}\} \\ &F_1 \leq F_2 \Leftrightarrow \bar{F}_1 \subseteq \bar{F}_2 \end{aligned}$$

For some sketches for explaining the geometrical construction for obtainin $W_R(P)$ see Figure 3.I - Figure 3.IV. Note, that we can consider $W_R(P)$ as a polyhedral complex.

We show some properties of the elements of $W_R(P)$:

Lemma 3.12. *Let $W_R(P)$ like defined above. Then it holds for $F \in W_R(P)$ that F is a face of $h(P)$ regarded as a polyhedra. Furthermore, $W_R(P)$ contains exactly the facets of $h(P)$, which are not contained in any hyperplane separating $h(P)$ and $h(-R)$.*

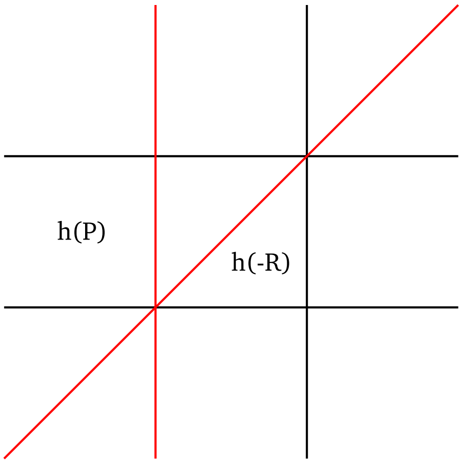


Figure 3.I: The arrangement $\mathcal{B}$ consists of the black hyperplanes and the arrangement $\mathcal{G}$ of the red ones.

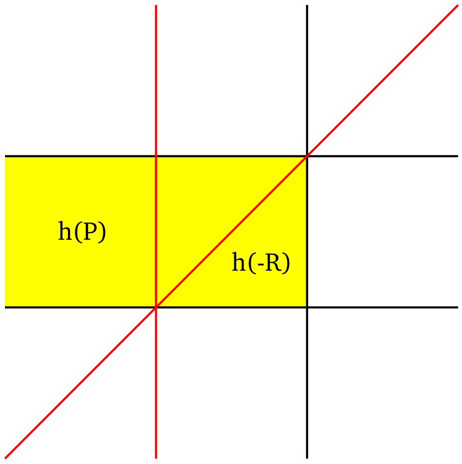


Figure 3.II: The region $B \in \mathcal{P}(\mathcal{B})$ which containes $h(P)$ and $h(-R)$ is the yellow one.

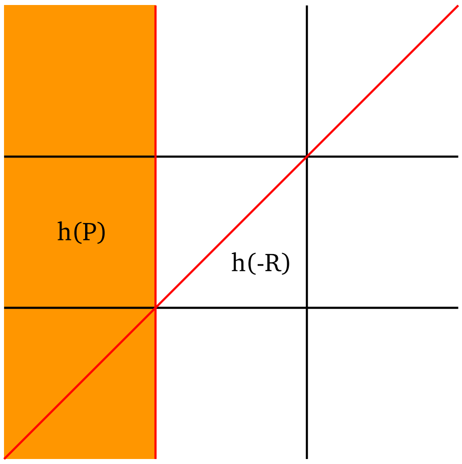


Figure 3.III: The region $G \in \mathcal{P}(\mathcal{G})$ which containes $h(P)$ is the orange one.

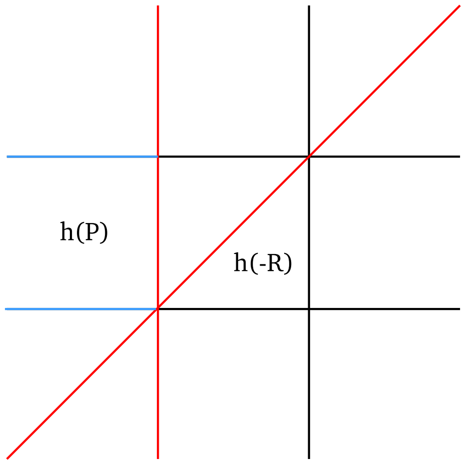


Figure 3.IV: The intersection of $\partial\bar{B}$ and $\bar{G}$ consists of the blue marked faces, namely two facets and two faces of dimension 0. This are the elements of $W_R(P)$.

Proof. Let everything like in Definition 3.11. Orient the hyperplanes such that for the sign vector of the region $h(P)$ it holds that

$$(h(P))_H = + \text{ for } H \in \mathcal{A}_{Z_d^{-(R)_d}} .$$

Therefore, for $B \in \mathcal{P}(\mathcal{B})$ it holds that

$$(B)_H = + \text{ for } H \in \mathcal{B}$$

and for $G \in \mathcal{P}(\mathcal{G})$ it holds that

$$(G)_H = + \text{ for } H \in \mathcal{G} .$$

Let $F \in \mathcal{F}(\mathcal{A})$ such that $F \subseteq \bar{G} \cap \partial \bar{B}$. Therefore, it holds that

$$(F)_H = + \text{ or } 0 \text{ for } H \in \mathcal{G}$$

and

$$(F)_H = + \text{ or } 0 \text{ for } H \in \mathcal{B} .$$

Since $\mathcal{A}_{Z_d^{-(R)_d}} = \mathcal{B} \cup \mathcal{G}$, we get that the entries of the sign vector of F are only 0 or $+$. Thus, F has to be a face of $h(P)$ considered as a polyhedra.
Let F be a facet of $h(P)$, which is not contained in any hyperplane separating $h(P)$ and $h(-R)$. Then F is contained in a hyperplane $H \in \mathcal{B}$. Since $h(P) \subseteq B$, we get $F \subseteq \bar{B}$. Otherwise since $h(P) \subseteq G$ we have $F \subseteq G$. Together, we get $F \in\subseteq \bar{G} \cap \partial \bar{B}$.
Let F be a facet of $h(P)$, which is contained in a hyperplane separating $h(P)$ and $h(-R)$, namely a hyperplane $H \in \mathcal{G}$. Then F is not contained in any hyperplane $H \in \mathcal{B}$ since $\mathcal{G}$ and $\mathcal{B}$ are disjoint by definition. Therefore, it holds that $F \not\subseteq \partial \bar{B}$ and hence we get $F \not\subseteq \bar{G} \cap \partial \bar{B}$. $\square$

Let $||W_R(P)|| := \bigcup_{F \in W_R(P)} F$. Then $||W_R(P)||$ is the underlying space of the polyhedral complex $W_R(S)$ and therefore we can compute its topological invariants, e.g. its reduced Euler characteristic. By Theorem 22.2 in [11] it holds that the topological Euler characteristic of $||W_R(P)||$ equals the Euler characteristic of the polyhedral complex $W_R(P)$.

Theorem 3.13. *Let $\mathcal{A} := \{H_1, \ldots, H_r\}$ be a central arrangement and $R \in \mathcal{P}(\mathcal{A})$, $P \in \mathcal{P}_{-(R)_d}(\mathcal{A})$. Then for the Möbius function of the reduced Edelman poset $\mathcal{EP}(\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\}, R)$ it holds that*

$$\mu_R(\hat{0}, P) = \tilde{\chi}(||W_R(P)||) = \tilde{\chi}(W_R(P)),$$

where $\tilde{\chi}(\cdot)$ denotes the reduced Euler characteristic.

This theorem is stated without proof in [7] where a proof is announced, but it never appeard. The following lemmata will fill in this gap and lead to a proof.

In the following, we use some intervals of the Edelman poset $\mathcal{EP}(\mathcal{P}(\mathcal{A}_{Z_d^{-(R)_d}}) \cup \{\hat{0}\}, R)$ and the Edelman poset $\mathcal{EP}(\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\}, R)$:

$$(\hat{0}, h(P))_{\mathcal{EP}(\mathcal{P}(\mathcal{A}_{Z_d^{-(R)_d}})\cup\{\hat{0}\},R)} \text{ and } (\hat{0}, P)_{\mathcal{EP}(\mathcal{P}_{-(R)_d}(\mathcal{A})\cup\{\hat{0}\},R)}$$

Since the two posets are isomorphic, we use the region R for indexing in both cases to improve the readability. If the upper bound of the interval is $h(P)$, we write

$$(\hat{0}, h(P))_R \text{ instead of } (\hat{0}, h(P))_{\mathcal{EP}(\mathcal{P}(\mathcal{A}_{Z_d^{-(R)_d}})\cup\{\hat{0}\},R)}$$

and if the upper bound is P, we write

$$(\hat{0}, P)_R \text{ instead of } (\hat{0}, P)_{\mathcal{EP}(\mathcal{P}_{-(R)_d}(\mathcal{A})\cup\{\hat{0}\},R)} \ .$$

We start our preliminary work for the proof of Theorem 3.13 by defining a helpful map:

Definition 3.14. Let $P \in \mathcal{P}_{-(R)_d}(\mathcal{A})$, $h(P) \in \mathcal{P}(\mathcal{A}_{Z_d^{-(R)_d}})$ and

$$\mathcal{C}_{h(P)} := \{H \in \mathcal{A}_{Z_d^{-(R)_d}} \mid H \cap \overline{h(P)} \text{ is a facet of } h(P)\} \ .$$

Define the follwing map:

$$f_{h(P)}: \begin{array}{rcl} (\hat{0}, h(P))_R & \to & W_R(P) \\ (f_{h(P)}(C))_i & = & \begin{cases} -(C)_i, & \text{if } H_i \in T(C, h(P)) \text{ and } H_i \notin \mathcal{C}_{h(P)} \\ 0, & \text{if } H_i \in T(C, h(P)) \text{ and } H_i \in \mathcal{C}_{h(P)} \\ C_i, & \text{if } H_i \notin T(C, h(P)) (\Leftrightarrow (C)_i = (h(P))_i) \end{cases} \end{array}$$

We show that $f_{h(P)}$ is well-defined. We recall some definitions:

$$\begin{aligned} &\mathcal{G} := T(h(P), h(-R)) \\ &G \in \mathcal{P}(\mathcal{G}) \text{ such that } h(P) \subseteq G \\ &\mathcal{B} := \mathcal{A}_{Z_d^{-(R)_d}} \setminus \mathcal{G} \\ &B \in \mathcal{P}(\mathcal{B}) \text{ such that } h(P), h(-R) \subseteq B \end{aligned}$$

Since either $(f_{h(P)}(C))_i = (P)_i$ or $(f_{h(P)}(C))_i = 0$ for some $H_i \in \mathcal{C}_{h(P)}$, we get $f_{h(P)}(C)$ is a face of $h(P)$. Hence $f_{h(P)}(C) \subseteq G$.
Since $T(h(P), h(-R)) \cap T(C, h(P)) = \emptyset$ and $f_{h(P)}(C)_i = 0$ for $H_i \in T(C, h(P)) \cap \mathcal{C}_{h(P)}$ we get that $f_{h(P)}(C)$ is a face of $h(P)$ which is only cut by regions which do not separate $h(P)$ and $h(-R)$. Hence $f_{h(P)}(C) \subset \partial\bar{B}$. Together we obtain $f_{h(P)}(C) \in \mathcal{F}(h(P))$ and $f_{h(P)}(C) \subseteq \partial\bar{B} \cap \bar{G}$, in other words $f_{h(P)}(C) \in W_R(P)$.

Lemma 3.15. *$f_{h(P)}$ is an order preserving map.*

Proof. Let $C, D \in (\hat{0}, h(P))_R$, such that $D \leq C$. This means $T(R, h^{-1}(D)) \subseteq T(R, h^{-1}(C)) \subseteq T(R, P)$ which leads to $T(h^{-1}(C), P) \subseteq T(h^{-1}(D), P)$. Since $C, D, h(P) \in \mathcal{P}(\mathcal{A}_{Z_d^{-(R)_d}})$, $T(h^{-1}(C), P) = T(C, h(P))$ and $T(h^{-1}(D), P) = T(D, h(P))$ holds. Because of the definition of $f_{h(P)}$ it holds that

$$\begin{aligned}(f_{h(P)}(C))_i = (f_P(D))_i \text{ for } H_i \in T(C, h(P)) \text{ , since } T(C, h(P)) \subseteq T(D, h(P)) \text{ ,}\\ \text{and}\\ (f_{h(P)}(C))_i = (f_P(D))_i \text{ for } H_i \in \mathcal{A}_{Z_d^{-(R)_d}} \setminus T(D, h(P)) \text{ .}\end{aligned}$$

Let $H_i \in T(D, h(P)) \setminus T(C, h(P))$. We have to distinguish the following cases:

Case 1: Let $H_i \in \mathcal{C}_{h(P)}$.
Then $(f_{h(P)}(D))_i = 0$ holds.

Case 2: Let $H_i \notin \mathcal{C}_{h(P)}$.
Then $(f_{h(P)}(D))_i = -(D)_i = (h(P))_i = (C)_i = (f_{h(P)}(C))_i$ since $H_i \notin T(C, h(P))$.

Hence either $(f_P(D))_i = (f_P(C))_i$ or $(f_P(D))_i = 0$ holds. This is equivalent to $f_P(D) \leq f_P(D)$, which means f_P is an order-preserving map. ☐

Lemma 3.16. *The fibres $f_P^{-1}(Q_{\leq F})$ are contractible for all $F \in W_R(P)$.*

Proof. It is sufficient to show, that every fibre has a unique maximal element (see [21]). First we show that $f_{h(P)}^{-1}(F)$ has a unique maximal element for every $F \in W_R(P)$. We distinguish the two cases:

Case 1: Let $(F)_i = 0$ for $H_i \notin \mathcal{C}_{h(P)}$ or $H_i \in T(h(P), h(-R))$.
Then $f_{h(P)}^{-1}(F) = \emptyset$. For details see the argumentation for showing that $f_{h(P)}$ is well-defined.

Case 2: Let $(F)_i = 0$ for $H_i \in \mathcal{C}_{h(P)}$ and $H_i \notin T(h(P), h(-R))$.
Then it holds that

$$(C_F)_i := \begin{cases} (P)_i, & \text{if } (F)_i \neq 0 \\ -(P)_i, & \text{if } (F)_i = 0 \end{cases}$$

is the unique maximal element of $f_{h(P)}^{-1}(F)$:
We check that $C_F \in f_{h(P)}^{-1}(F)$.
By definition it holds that, if $(F)_i = 0$, then $H_i \in T(C_F, h(P))$. Since $H_i \in \mathcal{C}_{h(P)}$ and $H_i \notin T(h(P), h(-R))$ holds by our case-by-case analysis, and
$T(C_F, h(P)) \cap T(h(P), h(-R)) = \emptyset$ holds, we get

$$(F)_i = 0 \Rightarrow H_i \in T(C_F, h(P)) \cap \mathcal{C}_{h(P)} \text{ .}$$

Here even $T(C_F, h(P)) \subset \mathcal{C}_{h(P)}$ holds. Hence $f_{h(P)}(C_F) = F$.
We check that C_F is the unique maximal element.
Let $C \in (\hat{0}, h(P))$ such that $C \in f_{h(P)}^{-1}(F)$ then $C \leq C_F$. Because of the definition of the ordering it is sufficient to show $(C)_i \neq (C_F)_i \Rightarrow (C)_i \neq (P)_i$. Let $(C)_i \neq (C_F)_i$.

(a) Let $(F)_i \neq 0$.
Then by definition $(C_F)_i = (P)_i$. Hence $(C)_i \neq (P)_i$ holds.

(b) Let $(F)_i = 0$.
Then by definition $(C_F)_i = -(P)_i$. Because we assume $(C)_i \neq (C_F)_i$ we have $(C)_i = (P)_i$. This means $H_i \notin T(C, h(P))$. By the definition of $f_{h(P)}$ follows $(f_{h(P)}(C))_i \neq 0$. This contradicts $(F)_i = 0$. Hence $(C)_i = (C_F)_i$ for $(F)_i = 0$.

We check that if $f_{h(P)}^{-1}(F) = \emptyset$ then $f_{h(P)}^{-1}(F') = \emptyset$ for every $F' \leq F$:
If $F' \leq F$ then either $F'_i = F_i$ or $F'_i = 0$. Hence, if $f_{h(P)}^{-1}(F) = \emptyset$, the argumentation of Case 1 holds for F', too, leading to $f_{h(P)}^{-1}(F') = \emptyset$.

We show if $f_{h(P)}^{-1}(F) \neq \emptyset$ then C_F is the unique maximal element of the fibre $f^{-1}(W_R(P)_{\leq F})$:
The following it holds, since f_P is an order-preserving map:

$$f^{-1}(W_R(P)_{\leq F}) = \bigcup_{F' \leq F} f_{h(P)}^{-1}(F') \ .$$

Assume, there is $C_{F'}$ such that $C_{F'} > C_F$. Then since $f_{h(P)}$ is order-preserving $f_{h(P)}(C_{F'}) = F' > F = f_{h(P)}(C_F)$ holds. This a contradiction to $F' < F$.
Assume, there is $C_{F'}$ such that $C_{F'}$ and C_F are incomparable. Hence $f_{h(P)}(C_{F'}) = F'$ and $f_{h(P)}(C_F) = F$ are incomparable, too. Again this contradicts $F' < F$.
Together each fibre has a unique maximal element and is therefore contractible.
□

Now we are in position to prove Theorem 3.13.

Proof of Theorem 3.13. Since $f_{h(P)} : (\hat{0}, h(P))_R \to W_R(P)$ is an order-preserving map (see Lemma 3.15) and the fibres $f_{h(P)}^{-1}(W_R(P)_{\leq F})$ are having a unique maximal element for all $F \in W_R(P)$ (see Lemma 3.16) we get by Quillen's fibre Lemma (Lemma 2.42):

$$||\Delta((\hat{0}, h(P))_R)|| \simeq ||\Delta(W_R(P))|| \ ,$$

in other words $||\Delta((\hat{0}, h(P))_R)||$ and $||\Delta(W_R(P))||$ are homotopy equivalent and therefore, they have the same reduced Euler characteristic. Since $||W_R(P)||$ is the underlying space of a regular CW-complex and $W_R(P)$ the corresponding poset we get by Lemma 2.46

$$||W_R(P)|| \text{ is homotopy equivalent to } ||\Delta(W_R(P))|| \ .$$

Hence,

$$||\Delta((\hat{0}, h(P))_R)|| \text{ is homotopy equivalent to } ||W_R(P)|| \ .$$

Since the closed interval $[\hat{0}, h(P)]$ is the bounded extension of the open interval $(\hat{0}, h(P))$ we get by Lemma 2.45

$$\mu_R(\hat{0}, h(P)) = \tilde{\chi}(\Delta((\hat{0}, h(P))_R)) \ .$$

Using the homotopy equivalences from above, it holds that

$$\mu_R(\hat{0}, h(P)) = \tilde{\chi}(||W_R(P)||) \ .$$

Since $(\hat{0}, h(P))_R$ is isomorphic to $(\hat{0}, P)_R$ we get

$$\mu_R(\hat{0}, P) = \tilde{\chi}(||W_R(P)||)$$

and Theorem 3.13 is shown. □

For our proof of Varchenko's Theorem for central arrangements (Theorem 3.5) we need some further properties of the Möbius-Function of the restriciton of Edelman's poset:

Theorem 3.17. *Let $\mathcal{A} := \{H_1, \ldots, H_r\}$ be a central arrangement of hyperplanes in $\mathbb{R}^n$. Let $H_d \in \mathcal{A}$ and $R \in \mathcal{P}(\mathcal{A})$, $P \in \mathcal{P}_{-(R)_d}(\mathcal{A})$.*

1. *If $h(P) \in \mathcal{P}(\mathcal{A}_{Z_d^{-(R)_d}})$ is bounded, then the following holds:*

$$\mu_R(\hat{0}, P) = \begin{cases} (-1)^{n-2} = (-1)^n & \textit{if } -R = P \\ 0 & \textit{otherwise} \end{cases} \tag{1}$$

2. *Let F denote the maximal face of R, which is contained in H_d. Then*

$$\mu_R(\hat{0}, P) = 0 \textit{ if } P \notin \textit{star}(F). \tag{2}$$

Proposition 5.5 in [7] resembles the first part of Theorem 3.17. The announced proof never appeared and unfortunately the second part of this proposition is not correct as stated.

We split the proof of Theorem 3.17 into two parts. By using the results of the previous sections, we are able to prove the first part.

Proof of the first part of Theorem 3.17.

If $h(P)$ is bounded, we can consider $h(P)$ as an $(n-1)$-polytope. If $P = -R$ we get $h(P) = h(-R)$, which leads to $||W_R(P)|| = \partial\overline{h(P)}$. Since the boundary complex of an $(n-1)$-polytope is a $(n-2)$-sphere, its reduced Euler characteristic is well-known and by using Theorem 3.13 we deduce

$$\mu_R(\hat{0}, P) = \tilde{\chi}(\partial\overline{h(P)}) = (-1)^{(n-1)-1} = (-1)^n \ .$$

If $P \neq -R$ we get $h(P) \neq h(-R)$. Hence $W_R(P)$ cannot be the whole boundary of $h(P)$ (see Figure 3.V) . It contains exactly the facets of $h(P)$, which are contained in hyperplanes separating $h(P)$ and $h(-R)$. Draw a line l from a point p inside $h(P)$ to a point in general position inside $h(-R)$. Thus this line crosses all the hyperplanes which are separating $h(P)$ and $h(-R)$. Especially it crosses the hyperplanes which contain a facet of $h(P)$ (see Figure 3.VI). We define a linear ordering on this facets: Let F, F' be facets of $h(P)$ which are not elements of $W_R(P)$ and let H, H' be the hyperplanes containing them. We define

$$F < F' \Leftrightarrow [p, l \cap H] \subset [p, l \cap H']$$

(see Figure 3.VII). By Theorem 2.26 we get, that this ordering is the beginning of a line shelling for the boundary complex of $h(P)$. By Lemma 2.27, the reverse order of the facets is a shelling, too. Therefore, we know that there exists a shelling of the boundary complex of $h(P)$ where facets of $h(P)$ which are not contained in $W_R(P)$ come last. Thus, there is a shelling of $\mathsf{C}(\partial \overline{h(P)})$, where the elements of $W_R(P)$ come first. By Lemma 2.29 it holds that

$$\tilde{\chi}(||W_R(P)||) = 0 \ .$$

Therefore, we get by Theorem 3.13

$$\mu_R(\hat{0}, P) = 0 \ .$$

□

For getting in position to prove the second part of Theorem 3.17 we have to prove some lemmata.
We recall the definition of a gated metric space and adapt it afterwards to our situation.

Definition 3.18 (see [8])**.** A subset $\mathcal{N}$ of a metric space $(\mathcal{M}, d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_{\geq 0})$ is called gated (in $\mathcal{M}$) if the following holds:
For every $C \in \mathcal{M}$, there exists a $C' \in \mathcal{N}$ such that

$$d(C, D) = d(C, C') + d(C', D)$$

for all $D \in \mathcal{N}$.

Remark 3.19. One can define a metric d on the space of regions of the arrangement $\mathcal{A}$ by $d := \#T(C, D)$ for $C, D \in \mathcal{P}(\mathcal{A})$. Let $F \in \mathcal{F}(\mathcal{A})$. Then $star(F)$ is called gated, if for every $C \in \mathcal{P}(\mathcal{A})$ exists a $C' \in star(F)$ such that

$$d(C, D) = d(C, C') + d(C', D)$$

for all $D \in star(F)$. This is equivalent to

$$T(C, D) = T(C, C') \dot{\cup} T(C', D) \ .$$

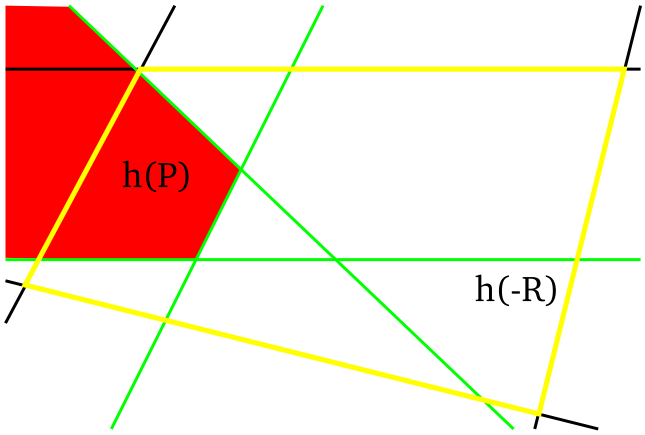


Figure 3.V: $||W_R(s)||$ equals the intersection of the closure of the red region of the arrangement consisting of the green hyperplanes and the yellow boundary of the region containing $h(P)$ and $h(-R)$ in the arrangement of the black hyperplanes.

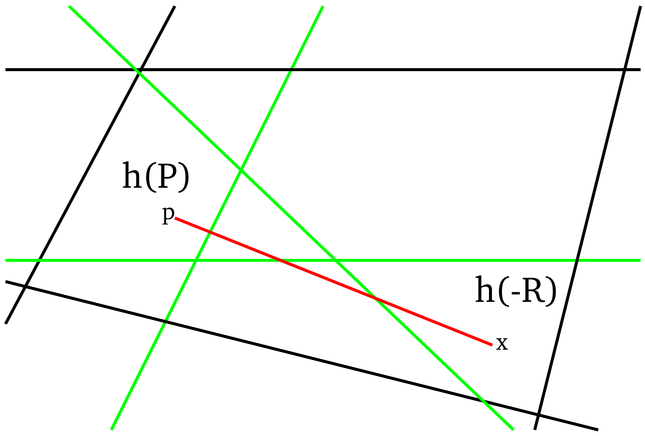


Figure 3.VI: The construction of the line for the line shelling.

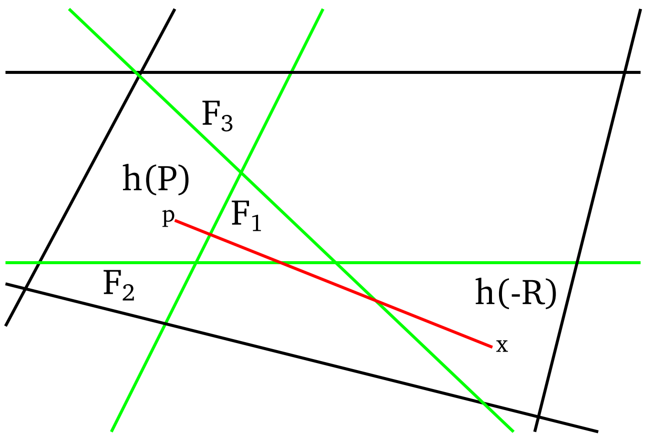


Figure 3.VII: By using the construction for a line shelling we obtain a shelling of the boundary complex of $h(P)$ where the facets which are not contained in $W_R(P)$ come first.

In the sequel we recall and extend some results around the gate property of arrangements. We refer to [1, p.4] and [8] :

Lemma 3.20. *Let $\mathcal{A}$ be a not necessarily central arrangement of hyperplanes and $F \in \mathcal{F}(\mathcal{A})$. Then for all $C \in \mathcal{P}(\mathcal{A})$ exists a unique $C' \in star(F)$, such that $T(C,D) = T(C,C')\dot{\cup}T(C',D)$ for all regions $D \in star(F)$.*

Proof. Show the existence of C':
Let $C \in \mathcal{P}(\mathcal{A})$ and $F \in \mathcal{F}(\mathcal{A})$. We distinguish between the following two cases.

- $C \in star(F)$:
  Since $C \in star(F)$ the equality $T(C,D) = T(C,C)\dot{\cup}T(C,D)$ holds for every $D \in star(F)$. Hence we can define $C' := C$ and we are done in this case.

- $C \notin star(F)$:
  Let $D \in star(F)$. By the product $F \circ C$ we get a region, which is contained in $star(F)$.
  Show that $F \circ C \in star(F)$ satisfies $T(C,D) = T(C, F \circ C)\dot{\cup}T(F \circ C, D)$ for all $D \in star(F)$: Assume $T(C, F \circ C) \cap T(F \circ C, D) \neq \emptyset$. This means, there exists a hyperplane $H \in \mathcal{A}$, such that H separates C and $F \circ C$ and H separates $F \circ C$ and D.
  Since H separates C and $F \circ C$, it cannot separate $F \circ C$ and $F \circ (-C)$:

$$\begin{aligned} H \in T(C, F \circ C) &\Rightarrow C_H = -(F \circ C)_H && (3)\\ &\Rightarrow C_H = -F_H (\text{particularly } F_H \neq 0) && (4) \end{aligned}$$

  Hence

$$\begin{aligned} (F \circ (-C))_H &= F_H \\ &= -(C)_H && \text{using Equation(4)} \\ &= -(-(F \circ C)_H) && \text{using Equation(3)} \\ &= (F \circ C)_H \end{aligned}$$

  This means, H does not separate $F \circ C$ and $F \circ (-C)$.
  Since $star(F) = [F \circ C, F \circ (-C)]$ and $D \in star(F)$, H does not separate $F \circ C$ and D. This is a contradiction to our assumption. Altogether, we get $T(C, F \circ C) \cap T(F \circ C, D) = \emptyset$, which implies $T(C,D) = T(C, F \circ C)\dot{\cup}T(F \circ C, D)$ for all $D \in star(F)$. We define $C' := F \circ C$.

Show the uniqueness of C':(see [8])
Let $C \in \mathcal{P}(\mathcal{A})$ and $F \in \mathcal{F}(\mathcal{A})$. Assume, there are two gates $C', C'' \in star(F)$. Using the gate property we get

$$T(C,C') = T(C,C'')\dot{\cup}T(C'',C') \quad (5)$$
$$T(C,C'') = T(C,C')\dot{\cup}T(C',C'') \ . \quad (6)$$

Inserting Equation (6) in Equation (5)

$$\begin{aligned} & T(C,C') &= T(C,C')\dot{\cup}T(C',C'')\dot{\cup}T(C'',C') \\ \Rightarrow\ & T(C',C'') &= \emptyset \end{aligned}$$

The last equation is equivalent to $C' = C''$. Altogether, the gate C is unique. □

Remark 3.21. We assume the notation of Lemma 3.20. Then

$$\#T(C,D) = \#T(C,C') + \#T(C',D) \text{ for all } D \in \mathcal{A}\ .$$

Hence

$$\#T(C,D) > \#T(C,C') \text{ for all } D \in \mathcal{A} \text{ such that } D \neq C'\ .$$

In other words: C' is the closest region to C, lying in $star(F)$.

Now we are in position to prove the second part of Theorem 3.17. First, we recall this part:

Theorem 3.17 [second part]. *Let $\mathcal{A} := \{H_1, \dots, H_r\}$ a central arrangement, Let $H_d \in \mathcal{A}$, $R \in \mathcal{P}(\mathcal{A})$, $P \in \mathcal{P}_{-(R)_d}(\mathcal{A})$ and let F denote the maximal face of R which is contained in H_d. Then*

$$\mu_R(\hat{0}, P) = 0, \text{ if } P \notin star(F)\ .$$

Proof of the second part of Theorem 3.17.
We show this part of Theorem 3.17 by induction on the rank. Let $P \in \mathcal{P}_{-(R)_d}(\mathcal{A})$ such that $P \notin star(F)$ and that there exists $Q \in star(F) \cap \mathcal{P}_{-(R)_d}(\mathcal{A})$, such that $Q < P$ is a covering. Because of the gate property (see Lemma 3.20) the region Q is unique. Using the definition of the Möbius function we calculate $\mu_R(\hat{0}, P)$.

$$\begin{aligned} \mu_R(\hat{0}, P) &= -\sum_{\hat{0} \leq S < P} \mu_R(\hat{0}, S) \\ &= -\mu_R(\hat{0}, Q) - \sum_{\hat{0} \leq S < Q} \mu_R(\hat{0}, S) \\ &= -\mu_R(\hat{0}, Q) - (-\mu_R(\hat{0}, Q)) \\ &= 0 \end{aligned}$$

Let $P \in \mathcal{P}_{-(R)_d}(\mathcal{A})$ such that $P \notin star(F)$ and that $Q < P$ is not a covering for any $Q \in star(F) \cap \mathcal{P}_{-(R)_d}(\mathcal{A})$. Let $P' < P$ such that $P' \notin star(F) \cap \mathcal{P}_{-(R)_d}(\mathcal{A})$. Then by induction hypothesis $\mu_R(\hat{0}, P') = 0$ and P' can be removed from the poset without any influence on $\mu_R(\hat{0}, P)$ (see Lemma 2.50). Hence we can remove all $P' < P$ such that $P' \notin star(F) \cap \mathcal{P}_{-(R)_d}(\mathcal{A})$. Because of the gate property, there is only one (unique) region $Q \in star(F) \cap \mathcal{P}_{-(R)_d}(\mathcal{A})$, such that $Q < P$ is a covering in the poset after the removals. Now a calculation similar to the one in the induction basis shows that $\mu_R(\hat{0}, Q) = 0$ holds. Thus, the assumption is proved. □

3.3 A Factorisation of Varchenko's Matrix

In this section our aim is to describe and prove a factorisation of Varchenko's Matrix $B(\mathcal{A})$ in the case of a central arrangement $\mathcal{A}$. The following lemma (see [7, p. 169]) is needed for calculating this factorisation. Here we adopt this lemma from its more general formulation to our situation.
In the following we use that each hyperplane $H_d \in \mathcal{A}$ separates $\mathbb{R}^n$ into two half-spaces, H_d^+ and H_d^-. We define

$$\mathcal{P}_{H_d^-}(\mathcal{A}) := \{P \in \mathcal{P}(\mathcal{A}) \mid (P)_d = -\}$$

$$\mathcal{P}_{H_d^+}(\mathcal{A}) := \{P \in \mathcal{P}(\mathcal{A}) \mid (P)_d = +\} \ .$$

Let $R \in \mathcal{P}_{H_d^+}(\mathcal{A})$. Then we are able to define the Edelman poset

$$\mathcal{EP}(\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\}, R) \ .$$

Lemma 3.22. *Let $H_d \in \mathcal{A}$ and $\mathcal{P}_{H_d^-}(\mathcal{A}), \mathcal{P}_{H_d^+}(\mathcal{A})$ like defined above. Let $B_{\mathcal{P}_{H_d^-}(\mathcal{A}),\mathcal{P}_{H_d^+}(\mathcal{A})}$ denote the matrix with entries $B(P,R)$ for $P \in P_{H_d^-}(\mathcal{A})$, $R \in \mathcal{P}_{H_d^+}(\mathcal{A})$ and $B_{P_{H_d^-}(\mathcal{A}),\mathcal{P}_{H_d^-}(\mathcal{A})}$ the matrix with entries $B(Q,S)$ for $Q,S \in \mathcal{P}_{H_d^-}(\mathcal{A})$. Then the following holds:*

$$B_{\mathcal{P}_{H_d^-}(\mathcal{A}),\mathcal{P}_{H_d^+}(\mathcal{A})} = B_{\mathcal{P}_{H_d^-}(\mathcal{A}),\mathcal{P}_{H_d^-}(\mathcal{A})} \cdot U \tag{7}$$

where U is a $l \times l$-matrix for $l := \#\mathcal{P}_{H_d^-}(\mathcal{A}) = \#\mathcal{P}_{H_d^+}(\mathcal{A})$ with entries

$$U_{P,R} = -\mu_R(\hat{0}, P) \cdot B(P,R) \tag{8}$$

where $P \in \mathcal{P}_{H_d^-}(\mathcal{A})$ and $R \in \mathcal{P}_{H_d^+}$ and $\mu_R(\cdot,\cdot)$ is the Möbius function of the poset $\mathcal{EP}(\mathcal{P}_{(-R)_d}(\mathcal{A}) \cup \{\hat{0}\}, R)$.

For proving Lemma 3.22, we need the following lemmata:

Lemma 3.23. *Let $R \in \mathcal{P}_{H_d^+}(\mathcal{A})$ and $P \in \mathcal{P}_{H_d^-}(\mathcal{A})$. Define a new arrangement of hyperplanes $\mathcal{A}'$ by $\mathcal{A}' := \mathcal{A} \setminus T(R,P)$. Let π denote the map from $\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\}$ to $\mathcal{P}(\mathcal{A}') \cup \{\hat{0}\}$ which is given by*

$$\pi : \begin{array}{ccc} \mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\} & \to & \mathcal{P}(\mathcal{A}') \cup \{\hat{0}\} \\ P & \mapsto & \mathsf{O} \text{ such that } P \subseteq \mathsf{O} \\ \hat{0} & \mapsto & \hat{0} \end{array} \ .$$

Then $\pi(\mathcal{P}_{-(R)_d} \cup \{\hat{0}\})$ with the order inherited from $\mathcal{EP}(\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\}, R)$ is a poset.

Proof. The assertion is clear, since $\mathcal{EP}(\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\}, R)$ is a poset and we consider the image of its ground set $\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\}$. ☐

We define the following objects:

Definition 3.24. Let $R \in \mathcal{P}_{H_d^+}(\mathcal{A})$ and $P \in \mathcal{P}_{H_d^-}(\mathcal{A})$. Then we define a new arrangement

$$\mathcal{A}' := \mathcal{A} \setminus T(R, P) \ .$$

Sometimes we have to distinguish between hyperplanes in $\mathcal{A}$ and hyperplanes in $\mathcal{A}'$ which are separating some regions. Therefore, we define

Definition 3.25. Let $\mathcal{A}$ be an arrangement of hyperplanes and $\mathcal{D} \subseteq \mathcal{A}$ an arbitrary subarrangement. Let $P, Q \in \mathcal{P}(\mathcal{A})$. We define

$$\begin{aligned} T_{\mathcal{A}}(P, Q) &:= \{H \in \mathcal{A} \mid H \text{ separates } P \text{ and } Q\} \\ T_{\mathcal{D}}(P, Q) &:= \{H \in \mathcal{D} \mid H \text{ separates } P \text{ and } Q\} \ . \end{aligned}$$

If the index is omitted, we always refer to the arrangement $\mathcal{A}$, meaning $T(P, Q) = \{H \in \mathcal{A} \mid H \text{ separates } P \text{ and } Q\}$.

Lemma 3.26. *Let $R \in \mathcal{P}_{H_d^+}(\mathcal{A})$ and $P, Q \in \mathcal{P}_{H_d^-}(\mathcal{A})$. Let $\mathcal{A}' := \mathcal{A} \setminus T(R, P)$. Then*

$$T_{\mathcal{A}'}(Q, R) = T_{\mathcal{A}}(P, Q) \cap T_{\mathcal{A}}(Q, R)$$

holds.

Proof. In the following π denotes the map from Lemma 3.23.
showing "$\subseteq$":
Let $H \in T_{\mathcal{A}'}(Q, R)$. Then H is separating $\pi(Q)$ and $\pi(R)$ in $\mathcal{A}'$.

$$\Rightarrow \pi(Q)_H = Q_H = -R_H = -\pi(R)_H$$

Since $\mathcal{A}' = \mathcal{A} \setminus T_{\mathcal{A}}(P, R)$, $H \notin T_{\mathcal{A}}(P, R)$.

$$\Rightarrow P_H = -R_H$$

Together we get

$$P_H = -Q_H$$

which is equivalent to $H \in T_{\mathcal{A}}(P, Q)$. Since $T_{\mathcal{A}'}(Q, R) \subseteq T_{\mathcal{A}}(Q, R)$, we get $H \in T_{\mathcal{A}}(Q, R) \cap T_{\mathcal{A}}(P, Q)$.
showing "$\supseteq$":
Let $H \in \mathcal{A}$ and $H \in T_{\mathcal{A}}(P, Q) \cap T_{\mathcal{A}}(Q, R)$. This means H separates P and Q and H separates Q and R:

$$\begin{aligned} P_H &= -Q_H \\ Q_H &= -R_H \\ \Rightarrow P_H &= R_H \end{aligned}$$

Hence, $H \notin T_{\mathcal{A}}(P, R)$. Since $\mathcal{A}' = \mathcal{A} \setminus T_{\mathcal{A}}(P, R)$ and $H \in T_{\mathcal{A}}(Q, R)$ we get $H \in T_{\mathcal{A}'}(Q, R)$. ∎

In the following, we need a special set of regions:

Definition 3.27. Let $R \in \mathcal{P}_{H_d^+}(\mathcal{A})$, $P \in \mathcal{P}_{H_d^-}(\mathcal{A})$ and $\mathcal{S} \subseteq \mathcal{A}$ a set of hyperplanes. We define the following set of regions:

$$\mathcal{P}(\mathcal{A})_{P,R}(\mathcal{S}) := \{Q \in \mathcal{P}_{-(R)_d}(\mathcal{A}) \mid T(P,Q) \cap T(Q,R) = \mathcal{S}\} \ .$$

Lemma 3.28. *Let $R \in \mathcal{P}_{H_d^+}(\mathcal{A})$ and $Q, Q' \in \mathcal{P}_{-(R)_d}(\mathcal{A})$. Let the map π from Lemma 3.23. Then the following holds:*

$$\pi(Q) = \pi(Q') \Leftrightarrow \textit{There exists } \mathcal{S} \subseteq \mathcal{A} \textit{ such that } Q, Q' \in \mathcal{P}(\mathcal{A})_{P,R}(\mathcal{S}) \ .$$

Proof.
showing "⇒":
Let $Q, Q' \in \mathcal{P}_{-(R)_d}(\mathcal{A})$ such that $\pi(Q) = \pi(Q')$. This means Q and Q' are mapped onto the same region of $\mathcal{A}' := \mathcal{A} \setminus T_{\mathcal{A}}(P,R)$. Hence, Q and Q' are only separated by hyperplanes contained in $T_{\mathcal{A}}(P,R)$. Therefore,

$$\begin{aligned}
&\Rightarrow T_{\mathcal{A}'}(Q,R) = T_{\mathcal{A}'}(Q',R) && \\
&\Rightarrow T_{\mathcal{A}}(P,Q) \cap T_{\mathcal{A}}(P,R) = T_{\mathcal{A}}(P,Q') \cap T_{\mathcal{A}}(P,R) && \text{Lemma 3.26} \\
&\Rightarrow Q, Q' \in \{Q'' \in \mathcal{P}_{H_d^-}(\mathcal{A}) : T_{\mathcal{A}}(P,Q'') \cap T_{\mathcal{A}}(Q'',R) = \mathcal{S}\} && \mathcal{S} := T_{\mathcal{A}}(P,Q) \cap T_{\mathcal{A}}(Q,R) \\
&\Rightarrow Q, Q' \in \mathcal{P}(\mathcal{A})_{P,R}(\mathcal{S}),\ \mathcal{S} \subseteq \mathcal{A} &&
\end{aligned}$$

showing "⇐":
Let $Q, Q' \in \mathcal{P}(\mathcal{A})_{P,R}(\mathcal{S})$ for some $\mathcal{S} \subseteq \mathcal{A}$.

$$\begin{aligned}
&\Rightarrow Q, Q' \in \{Q'' \in \mathcal{P}_{H_d^-}(\mathcal{A}) \mid T_{\mathcal{A}}(P,Q'') \cap T_{\mathcal{A}}(Q'',R)\} && \text{Def. of } \mathcal{P}(\mathcal{A})_{P,R}(\mathcal{S}) \\
&\Rightarrow T_{\mathcal{A}}(P,Q) \cap T_{\mathcal{A}}(Q,R) = \mathcal{S} = T_{\mathcal{A}}(P,Q') \cap T_{\mathcal{A}}(Q',R) && \\
&\Rightarrow T_{\mathcal{A}'}(Q,R) = T_{\mathcal{A}'}(Q',R) && \text{Lemma 3.26} \\
&\Rightarrow \pi(Q)_H = \pi(Q')_H \text{ for } H \in \mathcal{A}' && \\
&\Rightarrow \pi(Q) = \pi(Q') &&
\end{aligned}$$

□

Remark 3.29. Assume the situation of Lemma 3.28. Then in particular it holds for $Q, Q' \in \mathcal{P}_{-(R)_d}(\mathcal{A})$ that

$$\pi(Q) = \pi(Q') \Leftrightarrow T(P,Q) \cap T(Q,R) = T(P,Q') \cap T(Q',R) \ .$$

Lemma 3.30. *Let $R \in \mathcal{P}_{H_d^+}(\mathcal{A})$ and let $\mathsf{O} \in \pi(\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\})$ as defined in Lemma 3.23. Then the fibre $\pi^{-1}(\mathsf{O})$ contains a unique maximal element.*

Proof. Let $Q, Q' \in \pi^{-1}(\mathsf{O})$, such that Q, Q' are incomparable (meaning $Q \not\leq Q'$ and $Q' \not\leq Q$).
Show, that there is $\tilde{Q} \in \pi^{-1}(\mathsf{O})$, such that $Q, Q' < \tilde{Q}$:
Since Q, Q' are incomparable

$$T_{\mathcal{A}}(R, Q) \not\subseteq T_{\mathcal{A}}(R, Q') \text{ and } T_{\mathcal{A}}(R, Q') \not\subseteq T_{\mathcal{A}}(R, Q) \ .$$

But Q and Q' are only separated by regions in $T(P, R)$. Hence, the region with the separating set $T_{\mathcal{A}}(R, Q') \cup T_{\mathcal{A}}(R, Q)$ is lying in O and

$$Q < \mathsf{O} \text{ and } Q' < \mathsf{O}$$

holds.
The proof of the uniqueness is analogous. $\square$

Definition 3.31. Let $R \in \mathcal{P}_{H_d^+}(\mathcal{A})$ and $\pi(\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\})$ as defined in Lemma 3.23. Define a map

$$i : \pi(\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\}) \to \mathcal{EP}(\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\}, R)$$

such that $\mathsf{O} \in \pi(\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\})$ is mapped onto the maximal $Q \in \mathcal{EP}(\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\}, R)$ such that $\pi(Q) = \mathsf{O}$.

Lemma 3.32. *Under the identification which is given by the map i, π (see Lemma 3.23) defines a closure operation on $\mathcal{EP}(\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\}, R)$.*

Proof. Let $Q, Q' \in \mathcal{EP}(\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\}, R)$.
We check the three conditions of Definition 2.47:
Check, that $Q \leq i(\pi(Q))$:
Let $\pi(Q) = \mathsf{O}$. Then we identify $\pi(Q)$ with the unique maximal element of $\pi^{-1}(\mathsf{O})$ (this exists because of Lemma 3.30). Therefore, $Q \leq i(\pi(Q))$ holds.
Check, that $Q \leq Q' \Rightarrow i(\pi(Q)) \leq i(\pi(Q'))$:
Let $Q \leq Q'$. This means $T_{\mathcal{A}}(R, Q) \subseteq T_{\mathcal{A}}(R, Q')$. Since $\mathcal{A}' = \mathcal{A} \setminus T_{\mathcal{A}}(P, R)$, we get

$$T_{\mathcal{A}'}(R, Q) = T_{\mathcal{A}}(R, Q) \setminus T_{\mathcal{A}}(P, R) \subseteq T_{\mathcal{A}}(R, Q') \setminus T_{\mathcal{A}}(P, R) = T_{\mathcal{A}'}(R, Q') \ ,$$

which is equivalent to $\pi(Q) := \mathsf{O} \leq \pi(Q') := \mathsf{O}'$. Because we identify O with the unique maximal element of $\pi^{-1}(\mathsf{O})$ and O' with the unique maximal element of $\pi^{-1}(\mathsf{O}')$, this leads to $i(\pi(Q)) \leq i(\pi(Q'))$.
Check, that $i(\pi(i(\pi(Q)))) = i(\pi(Q))$:
Let $\mathsf{O} := \pi(Q)$. We identify O with the unique maximal element of $\pi^{-1}(\mathsf{O})$. Name this element Q'. Hence $i(\pi(Q)) = Q'$. Applying π to Q' we get $\pi(Q') = \mathsf{O}$, since $Q' \subseteq O$. As Q' was defined as the maximal element of $\pi^{-1}(\mathsf{O})$, we get $i(\pi(i(\pi(Q)))) = i(\pi(Q')) = Q'$. Therefore, $i(\pi(i(\pi(Q)))) = i(\pi(Q))$ holds. $\square$

Now we are in position to prove Lemma 3.22:

Proof of Lemma 3.22. Let $P \in \mathcal{P}_{H_d^-}(\mathcal{A})$ and $R \in \mathcal{P}_{H_d^+}(\mathcal{A})$. By using matrix multiplication we get that the claim of Lemma 3.22 is equivalent to

$$\begin{aligned} B_{P,R} &= \sum_{Q \in \mathcal{P}_{H_d^-}(\mathcal{A})} B_{P,Q} U_{Q,R} \\ \Rightarrow B(P,R) &= \sum_{Q \in \mathcal{P}_{H_d^-}(\mathcal{A})} B(P,Q) \cdot (-\mu_R(\hat{0},Q) B(Q,R)) \\ &= \sum_{Q \in \mathcal{P}_{H_d^-}(\mathcal{A})} -\mu_R(\hat{0},Q) B(P,Q) B(Q,R) \end{aligned}$$

For all $Q \in \mathcal{P}_{H_d^-}(\mathcal{A})$ it holds that:

$$B(P,Q)B(Q,R) = B(P,R) \cdot \prod_{H \in T(P,Q) \cap T(Q,R)} a_H^2$$

We define $\mathcal{S} := T(P,Q) \cap T(Q,R)$. Using this equality, we get

$$B(P,R) = \sum_{Q \in \mathcal{P}_{H_d^-}(\mathcal{A})} -\mu_R(\hat{0},Q) B(P,R) \prod_{H \in \mathcal{S}} a_H^2$$

Hence, it is sufficient to show, that

$$\sum_{Q \in \mathcal{P}(\mathcal{A})_{P,R}(\mathcal{S})} \mu_R(\hat{0},Q) = \begin{cases} -1, & \text{if } \mathcal{S} = \emptyset \\ 0, & \text{otherwise} \end{cases}$$

Because of Lemma 3.32 we know, that under the identification (see Definition 3.31), the map π defines a closure operation on $\mathcal{EP}(\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\}, R)$. For improving the readability of the proof, we omit the map of the identification in the following. The poset $\pi(\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\})$ is isomorphic to the quotient of the closure operation π. Let $\mathsf{O} \in \pi(\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\})$.
Using Rota's Lemma (Lemma 2.49),

$$\begin{aligned} \mu_{\pi(\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\})}(\hat{0},\mathsf{O}) &= \sum_{Q \in \mathcal{EP}(\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\}, R), \pi(Q) = \mathsf{O}} \mu_R(\hat{0},Q) && \text{(Rota)} \\ &= \sum_{Q \in \mathcal{EP}(\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\}, R), Q \in \pi^{-1}(\mathsf{O})} \mu_R(\hat{0},Q) \\ &= \sum_{Q \in \mathcal{P}(\mathcal{A})_{P,R}(\mathcal{S})} \mu_R(\hat{0},Q) && \text{Remark 3.29} \end{aligned}$$

Recall, that the arrangement $\mathcal{A}'$ is defined by $\mathcal{A}' := \mathcal{A} \setminus T(R,P)$. Therefore, in the arrangement $\mathcal{A}'$, there is no hyperplane which separates P and R. Hence, $\pi(P)$ is the only element covering $\hat{0}$, which is the unique minimal element of the poset $\pi(\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\})$.

This leads to

$$\mu_{\pi(\mathcal{P}_{-(R)_d}(\mathcal{A})\cup\{\hat{0}\})}(\hat{0},\mathsf{O}) = \begin{cases} -1, & \text{if } \mathsf{O} = \pi(P) \\ 0, & \text{otherwise} \end{cases}.$$

Since $\emptyset = T_{\mathcal{A}'}(P,R) = T_{\mathcal{A}}(P,P) \cap T_{\mathcal{A}}(P,R)$, we get by Lemma 3.28 $\mathsf{O} = \pi(P) \Leftrightarrow \mathcal{S} = \emptyset$. Therefore,

$$\mu_{\pi(\mathcal{P}_{-(R)_d}(\mathcal{A})\cup\{\hat{0}\})}(\hat{0},\mathsf{O}) = \sum_{Q\in\mathcal{P}(\mathcal{A})_{P,R}(\mathcal{S})} \mu_R(\hat{0},Q) = \begin{cases} -1, & \text{if } \mathcal{S} = \emptyset \\ 0, & \text{otherwise} \end{cases}.$$

Because of our argumentation in the beginning of this proof, we are done. □

The next proposition, which can be found in [7, p.172], is the first step for obtaining the factorisation of Varchenko's Matrix $B(\mathcal{A})$.

Proposition 3.33. *Let $\mathcal{A} := \{H_1,\ldots,H_r\}$ be a central arrangement and $H_d \in \mathcal{A}$. Then*

$$B(\mathcal{A}) = B(\mathcal{A})[a_{H_d} = 0] \cdot M_{H_d} \tag{9}$$

where $B[a_{H_d} = 0]$ denotes the matrix B with weight a_{H_d} set to zero and M_{H_d} is a matrix with the following entries:

$$(M_{H_d})_{(P,R)} = \begin{cases} 1 & \textit{if } P = R \\ -\mu_R(\hat{0},P)B(\mathcal{A})_{(P,R)} & \textit{if } H_d \textit{ separates } P \textit{ and } R \\ 0 & \textit{otherwise} \end{cases} \tag{10}$$

where μ_R denotes the Möbius function of the poset $\mathcal{EP}(\mathcal{P}_{-(R)_d}(\mathcal{A}) \cup \{\hat{0}\}, R)$.

Remark 3.34. In the following proof we assume a certain order of the columns and rows of the matrix $B(\mathcal{A})$. Since we are interchanging rows and columns simultaneously, this change of basis does not effect the determinant of $B(\mathcal{A})$. As the calculation of this determinant is our main goal and for keeping the notation in the following proofs as simple as possible we will not perform the change of the basis by multiplying with the corresponding permutation matrices.

Proof. Let $\mathcal{P}(\mathcal{A})$ denote the set of regions of the arrangement $\mathcal{A}$ and define $n := \#\mathcal{P}(\mathcal{A})$. Let $H_d \in \mathcal{A}$. We define

$$\mathcal{P}_{H_d^-}(\mathcal{A}) := \{P \in \mathcal{P}(\mathcal{A}) \mid (P)_d = -\}$$

$$\mathcal{P}_{H_d^+}(\mathcal{A}) := \{P \in \mathcal{P}(\mathcal{A}) \mid (P)_d = +\}$$

Define a linear ordering on $\mathcal{P}(\mathcal{A})$ such that the $\frac{n}{2}$ elements of $\mathcal{P}_{H_d^-}(\mathcal{A})$ come first and that $P_{\frac{n}{2}+i} = -(P_i)$ holds:

$$P_1 < P_2 < \ldots < P_{\frac{n}{2}} < -(P_1) < -(P_2) < \ldots < -(P_{\frac{n}{2}})$$

Using this linear order for indexing the columns and rows of the matrix $B(\mathcal{A})$, we get

$$B(\mathcal{A}) = \left(\begin{array}{c|c} B_{\mathcal{P}_{H_d^-}(\mathcal{A}),\mathcal{P}_{H_d^-}(\mathcal{A})} & B_{\mathcal{P}_{H_d^-}(\mathcal{A}),\mathcal{P}_{H_d^+}(\mathcal{A})} \\ \hline B_{\mathcal{P}_{H_d^+}(\mathcal{A}),\mathcal{P}_{H_d^-}(\mathcal{A})} & B_{\mathcal{P}_{H_d^+}(\mathcal{A}),\mathcal{P}_{H_d^+}(\mathcal{A})} \end{array}\right)$$

Since

$$B(P, R) = B(-P, -R)$$

holds for all $P, R \in \mathcal{P}(\mathcal{A})$, we have

$$B_{\mathcal{P}_{H_d^-}(\mathcal{A}),\mathcal{P}_{H_d^+}(\mathcal{A})} = B_{\mathcal{P}_{H_d^+}(\mathcal{A}),\mathcal{P}_{H_d^-}(\mathcal{A})}$$
$$B_{\mathcal{P}_{H_d^-}(\mathcal{A}),\mathcal{P}_{H_d^-}(\mathcal{A})} = B_{\mathcal{P}_{H_d^+}(\mathcal{A}),\mathcal{P}_{H_d^+}(\mathcal{A})}$$

Using Lemma 3.22 and the symmetries in the matrix from above, we get

$$B(\mathcal{A}) = \underbrace{\left(\begin{array}{c|c} B_{\mathcal{P}_{H_d^-}(\mathcal{A}),\mathcal{P}_{H_d^-}(\mathcal{A})} & 0 \\ \hline 0 & B_{\mathcal{P}_{H_d^+}(\mathcal{A}),\mathcal{P}_{H_d^+}(\mathcal{A})} \end{array}\right)}_{:= B(\mathcal{A})[a_{H_d}=0]} \underbrace{\left(\begin{array}{c|c} I & U \\ \hline U & I \end{array}\right)}_{:= M_{H_d}}$$

where $B(\mathcal{A})[a_{H_d} = 0]$ denotes the matrix $B(\mathcal{A})$ with the weight a_{H_d} set to 0. This leads to Proposition 3.33. ☐

By applying this proposition to each hyperplane, we obtain the factorisation of Varchenko's Matrix which is described in the following theorem (see [7, p.173]):

Theorem 3.35. *Let $\mathcal{A} = \{H_1, \dots, H_r\}$ be a real, central arrangement of hyperplanes. Fix an ordering on the hyperplanes via $H_1 < \dots < H_r$. Then*

$$B = M_{H_1} \cdot M_{H_2} \cdot \dots \cdot M_{H_r}$$

where M_{H_d} *is the matrix over* $\mathbb{Z}[a_{H_1},\dots,a_{H_d}]$ *given by*

(i) $(M_{H_d})_{(P,P)} = 1$

(ii) $(M_{H_d})_{(P,R)} = -\mu_R(\hat{0},P)\cdot B(P,R)$ *if* $d = max\{k : H_k \in T(P,R)\}$

(iii) $(M_{H_d})_{(P,R)} = 0$ *otherwise*

In [7] is only said, that the theorem is reached by applying Proposition 3.33 to each hyperplane. For the sake of completeness we give a detailed proof.

Proof. We fix a linear order $H_1 < H_2 < \ldots < H_r$ on the hyperplanes of $\mathcal{A}$ and prove the theorem by induction on this linear order. Then by Proposition 3.33

$$B(\mathcal{A}) = B(\mathcal{A})[a_{H_r} = 0]\cdot M_{H_r}$$

holds. Using Proposition 3.33 successively, we get for $2 \le l \le r$

$$B(\mathcal{A})[a_{H_r} = \ldots = a_{H_i} = 0] = B(\mathcal{A})[a_{H_r} = \ldots = a_{H_i} = a_{H_{i-1}} = 0]\cdot M_{H_{i-1}}\ .$$

This leads to

$$\begin{aligned} B(\mathcal{A}) &= B(\mathcal{A})[a_{H_r} = 0]\cdot M_{H_r}\\ &= B(\mathcal{A})[a_{H_r} = a_{H_{r-1}} = 0]\cdot M_{H_{r-1}}\cdot M_{H_r}\\ &= \ldots\\ &= B(\mathcal{A})[a_{H_r} = \ldots = a_{H_2} = 0]\cdot M_{H_2}\cdot\ldots\cdot M_{H_r}\\ &= \underbrace{B(\mathcal{A})[a_{H_r} = \ldots = a_{H_1} = 0]}_{=Id}\cdot M_{H_1}\cdot\ldots\cdot M_{H_r}\\ &= M_{H_1}\cdot M_{H_2}\cdot\ldots\cdot M_{H_r}\ , \end{aligned}$$

which is the first part of Theorem 3.35.
For determining the entries of M_{H_i} for $1 \le i \le r$ we use Proposition 3.33 again (we define $B(\mathcal{A})[a_{H_{r+1}} = 0] := B(\mathcal{A})$). This leads to

$$(M_{H_i})_{(P,R)} = \begin{cases} 1 & \text{if } P = R\\ -\mu_R(\hat{0},P)\cdot B(\mathcal{A})[a_{H_r} = \ldots = a_{H_{i+1}} = 0]_{(P,R)} & \text{if } H_i \text{ separates}\\ & P \text{ and } R\\ 0 & \text{otherwise} \end{cases}$$

Let H_i separate P and R, i.e. $H_i \in T(P,R)$. Assume, $i < \max_k\{k : H_k \in T(P,R)\} := s$. Then, since we are applying Proposition 3.33 to the matrix $B(\mathcal{A})[a_{H_r} = \ldots = a_{H_{r-i}} = 0]$, $a_{H_s} = 0$. Hence $(M_{H_i})_{(P,R)} = B(\mathcal{A})[a_{H_r} = \ldots = a_{H_{i+1}} = 0]_{(P,R)} = 0$. If $i = \max_k\{k : H_k \in T(P,R)\}$, $B(\mathcal{A})[a_{H_r} = \ldots = a_{H_{i+1}} = 0]_{(P,R)} = B(P,R)$ holds, since all the weights corresponding to hyperplanes in $T(P,R)$ are not changed. Alltogether we get

$$(M_{H_i})_{(P,R)} = \begin{cases} 1 & \text{if } P = R\\ -\mu_R(\hat{0},P)\cdot B(P,R) & \text{if } i = \max_k\{k : H_k \in T(P,R)\}\\ 0 & \text{otherwise} \end{cases}$$

□

3.4 Some geometric properties

The following lemma will be used later on for calculating the Möbius function of certain Edelman Posets.

Lemma 3.36. *Let $\mathcal{A}$ be a central arrangement in $\mathbb{R}^n$ such that $\bigcap_{H\in\mathcal{A}} H = \{0\}$. Let $\hat{H} \in \mathcal{A}$ be a hyperplane, Z a parallel to $\hat{H}$ and $\mathcal{A}_Z$ the decone of $\mathcal{A}$ with respect to Z. Then the following holds:*

$$\hat{P} \in \mathcal{P}(\mathcal{A}_Z) \text{ is bounded } \Leftrightarrow \dim \hat{H} \cap \bar{P} = 0$$

Proof.
" $\Rightarrow$ ": Assume, $\hat{P} \in \mathcal{P}(\mathcal{A}_Z)$ is bounded. Then $\hat{P}$ can be written by a convex combination of its vertices $\hat{v}_1, \ldots, \hat{v}_r \in \mathbb{R}^{n-1}$:

$$\hat{P} = \left\{ \sum_{i=1}^{r} \lambda_i \hat{v}_i \mid \lambda_i \geq 0, \sum_{i=1}^{r} \lambda_i = 1 \right\}$$

Let $f_{\hat{H}}(x)$ denote the linear function, which is defining $\hat{H}$ by

$$\hat{H} := \{x \in \mathbb{R}^n \mid f_{\hat{H}}(x) = 0\}\ .$$

Since the arrangement $\mathcal{A}$ is deconed by a parallel to $\hat{H}$, we know that

$$f_{\hat{H}}(x) = a \text{ for } x \in \hat{P}$$

holds. By coning we can go back to the original arrangement $\mathcal{A}$. Then we get that P is a cone, which exterior rays are defined by the vectors $v_i \in \mathbb{R}^n$, $1 \leq i \leq n$, corresponding to $\hat{v}_i \in \mathbb{R}^{n-1}$, $1 \leq i \leq n$:

$$P = \left\{ \sum_{i=1}^{r} \lambda_i v_i \mid \lambda_i \geq 0 \right\}\ .$$

We show that $\hat{H} \cap \bar{P} = \{0\}$:
" $\subseteq$ " : Let $x \in \hat{H} \cap \bar{P} = \{0\}$. Then $f_{\hat{H}}(x) = 0$ and there exist $\lambda_i \geq 0$ such that $\sum_{i=1}^{r} \lambda_i v_i = x$. This leads to

$$\begin{aligned}
0 = f_{\hat{H}}(x) &= f_{\hat{H}}(\sum_{i=1}^{r} \lambda_i v_i) \\
&= \sum_{i=1}^{r} \lambda_i f_{\hat{H}}(v_i) && \text{linearity of } f_{\hat{H}}(x) \\
&= \sum_{i=1}^{r} \lambda_i a && \lambda_i \geq 0, a \neq 0 \\
&= a \sum_{i=1}^{r} \lambda_i
\end{aligned}$$

Hence, $\lambda_i = 0$ for $1 \leq i \leq r$ holds, which leads to

$$x = \sum_{i=1}^{r} 0 \cdot v_i = 0 \ .$$

Therefore, the inclusion holds.
" $\supseteq$ " : Since $f_{\hat{H}}(x)$ is a linear function, $f_{\hat{H}}(0) = 0$ holds. Hence $0 \in \hat{H}$.
Since $\lambda_i \geq 0$, $0 = \sum_{i=0}^{r} v_i \cdot 0 \in \bar{P}$. Thus the inclusion holds.
" $\Leftarrow$ ": Since P is a region of a central arrangement, P is a cone. Hence we can use the following for describing P:

$$P := \left\{ \sum_{i=1}^{r} \lambda_i v_i \mid \lambda_i \geq 0 \right\} \ ,$$

where the $v_i \in \mathbb{R}^n$ $1 \leq i \leq r$ define the exterior rays of P. Let $f_{\hat{H}}$ be the linear form such that $\hat{H} = \{x \in \mathbb{R}^n \mid f_{\hat{H}} = 0\}$ and let $\{x \in \mathbb{R}^n \mid f_{\hat{H}}(x) \geq 0\}$ be the half-space, which contains P.
Let $d > 0$. Define

$$S := \{x \in \mathbb{R}^n \mid f_{\hat{H}}(x) = d\} \cap P \subseteq \mathbb{R}^{n-1}$$

Since $d > 0$ $S \neq \emptyset$ holds. By using the description of P mentioned above we get

$$S = \left\{ \sum_{i=1}^{r} \lambda_i v_i \mid \lambda_i \geq 0,\ f_{\hat{H}} \left(\sum_{i=1}^{r} \lambda_i v_i \right) = d \right\}$$

Let $\sum_{i=1}^{r} \lambda_i v_i \in S$. By using the linearity of $f_{\hat{H}}$ we get

$$f_{\hat{H}} \left(\sum_{i=1}^{r} \lambda_i v_i \right) = \sum_{i=1}^{r} \lambda_i f_{\hat{H}}(v_i) = d$$

Since $f_{\hat{H}}(x) \geq 0$ for all $x \in P$ and $v_i \neq 0$ for $1 \leq i \leq r$, $f_{\hat{H}}(v_i) > 0$ holds for $1 \leq i \leq r$. Because of this fact and of the equation above we can deduce

$$0 \leq \lambda_i \leq \frac{d}{f_{\hat{H}}(v_i)} \text{ for } 1 \leq i \leq r$$

This leads to

$$S \subseteq \left\{ \sum_{i=1}^{r} \lambda_i v_i \mid \lambda_i \in \left[0, \frac{d}{f_{\hat{H}}(v_i)}\right] \right\} := M$$

Define a continuous map by

$$\varphi : \begin{array}{ccc} \mathbb{R}^r & \to & \mathbb{R}^n \\ (\lambda_1, \ldots, \lambda_r) & \mapsto & \sum_{i=1}^{r} \lambda_i v_i \end{array}$$

Then M is the image of the compact interval $\times_{i=1}^{r}\left[0, \frac{d}{f_{\hat{H}}(v_i)}\right]$:

$$M = \varphi\left(\times_{i=1}^{r}\left[0, \frac{d}{f_{\hat{H}}(v_i)}\right]\right)$$

Since the images of compact sets under continuous maps are compact, we conclude that M is compact. Hence in particular M is bounded. Since $S \subseteq M$, S is bounded, too.
As there is always a $d > 0$, such that $\hat{P} = S$, we have proved, that under the conditions in the lemma $\hat{P}$ is bounded. $\square$

We formulate another few lemmata which will be helpful for the proof of Theorem 3.5, the goal of this chapter.

Definition 3.37. Let $F \in \mathcal{F}(\mathcal{A})$ and $L \in \mathcal{E}(\mathcal{A})$ such that $F \subseteq L$ and $\dim(F) = \dim(L)$. Then we call L the continuation of F.

Lemma 3.38. *Let $F \in \mathcal{F}(\mathcal{A})$ be a face of the arrangement $\mathcal{A}$ and $L \in \mathcal{E}(\mathcal{A})$ be the continuation of F. Let $P_i \in \mathcal{P}(\mathcal{A})$ and $P_j = (P_i)_{(-F)}$ the image of P_i under the reflection in F. Then*

$$B(P_i, P_j) = a(L)$$

holds.

Proof. The lemma is proved by the definition of the weight of an intersection. $\square$

Let $H \in \mathcal{A}$ and Let $F \in \mathcal{F}(\mathcal{A})$ a face of the arrangement $\mathcal{A}$ such that $F \subseteq H_d$. We define the set

$$P_{H,F}(\mathcal{A}) := \{P \in \mathcal{P}(\mathcal{A}) \mid \bar{P} \cap H = \bar{F}\}\ .$$

Furthermore we define for $F \in \mathcal{F}(\mathcal{A})$:

$$star(F) := \{F' \in \mathcal{F}(\mathcal{A}) \mid \bar{F'} \cap F = F\}\ .$$

Lemma 3.39. *Let $H \in \mathcal{A}$ and $F, F' \in \mathcal{F}(\mathcal{A})$ such that $F, F' \subseteq H$. Then the following holds:*

$$\mathcal{P}_{H,F}(\mathcal{A}) \cap \mathcal{P}_{H,F'}(\mathcal{A}) = \emptyset \textit{ for } F \neq F' \tag{11}$$

$$\dot{\bigcup_{F \in \mathcal{F}(\mathcal{A})}} \mathcal{P}_{H,F}(\mathcal{A}) = \mathcal{P}(\mathcal{A}) \tag{12}$$

$$\textit{if } \dim(F) < \dim(F') \Rightarrow star(F') \cap \mathcal{P}_{H,F}(\mathcal{A}) = \emptyset \tag{13}$$

$$\textit{if } F \neq F' \textit{ and } \dim(F) = \dim(F') \Rightarrow star(F') \cap \mathcal{P}_{H,F}(\mathcal{A}) = \emptyset \tag{14}$$

Proof.
Part 1: Assume, there is $P \in \mathcal{P}_{H,F}(\mathcal{A}) \cap \mathcal{P}_{H,F'}(\mathcal{A})$ for $F \neq F'$. But then $\bar{P} \cap H = \bar{F}$ and $\bar{P} \cap H = \bar{F'}$ holds. This contradicts $F \neq F'$.
Part 2: The inclusion "$\subseteq$" and that the union is disjoint is clear by the definition of $\mathcal{P}_{H,F}(\mathcal{A})$. The other inclusion "$\supseteq$" holds, because - since $\mathcal{A}$ is central - $\bar{P} \cap H \neq \emptyset$ for every $P \in \mathcal{P}(\mathcal{A})$. Furthermore $\bar{P} \cap H$ is the closure of a face of $\mathcal{A}$. Therefore, it exists $F \in \mathcal{F}(\mathcal{A})$, such that $P \in \mathcal{P}_{H,F}(\mathcal{A})$.
Part 3: Let $\dim(F) < \dim(F')$. Assume, there is $P \in star(F') \cap \mathcal{P}_{H,F}(\mathcal{A})$. Then $F' = \bar{P} \cap F'$ and since $F' \subseteq H$ we get $F' \subseteq \bar{P} \cap H$. Otherwise it holds $\bar{P} \cap H = \bar{F}$. Hence $F' \subseteq \bar{F}$, which contradicts $\dim(F) < \dim(F')$.
Part 4: Let $F \neq F'$ and $\dim(F) = \dim(F')$. Assume, there is $P \in star(F') \cap \mathcal{P}_{H,F}(\mathcal{A})$ for $F \neq F'$. Since $F' \subseteq H$ we get $F' \subseteq \bar{P} \cap H$ by the definition of $star(F')$ and $\bar{F} = \bar{P} \cap H$ by the definition of $\mathcal{P}_{H,F}(\mathcal{A})$. Therefore, $F' \subseteq \bar{F}$ holds. Because of $\dim(F) = \dim(F')$ and since F and F' are faces this contradicts $F \neq F'$. □

Lemma 3.40. *Let $L \in \mathcal{E}(\mathcal{A})$ and $H \in \mathcal{A}$ such that $L \subseteq H$. Then*

$$\begin{aligned}&\{P \in \mathcal{P}(\mathcal{A}) \mid L \textit{ is the minimal intersection containing } \bar{P} \cap H\} \\ = &\dot{\bigcup_{\substack{F \subseteq H \\ L \textit{ is continuation of } F}}} \mathcal{P}_{H,F}(\mathcal{A}) \ .\end{aligned}$$

Therefore, the following holds

$$l_{\mathcal{P}(\mathcal{A})}(L) = \frac{1}{2} \sum_{\substack{F \subseteq H \\ L \textit{ is continuation of } F}} \#\mathcal{P}_{H,F}(\mathcal{A}) \ .$$

Proof. Let $L \in \mathcal{E}(\mathcal{A})$. Fix a hyperplane $H \in \mathcal{A}$ such that $L \subseteq H$. Let $P \in \mathcal{P}(\mathcal{A})$ such that L is the minimal intersection containing $\bar{P} \cap H$. This is equivalent to $\bar{F} := \bar{P} \cap H$ is the closure of a face F with $\dim(F) = \dim(L)$ and $F \subseteq L$. This intern is equivalent to L is the continuation of F and $P \in \mathcal{P}_{H,F}(\mathcal{A})$.
That the union is a disjoint union follows from the first part of Lemma 3.39. Using this and Definition 3.4 we get

$$l_{\mathcal{P}(\mathcal{A})}(L) = \frac{1}{2} \sum_{\substack{F \subseteq H \\ L \text{ is continuation of } F}} \#\mathcal{P}_{H,F}(\mathcal{A}) \ .$$

□

3.5 Proof of Varchenko's Theorem for central arrangements

Now we are in position to proof Varchenko's Theorem for central arrangements (Theorem 3.5). For the convenience of the reader we recall the theorem:

Theorem. *Let $\mathcal{A}$ be a central arrangement.*
Then for the determinant of the bilinear form $B(\cdot,\cdot)$ the following holds:

$$\det B(\mathcal{A}) = \prod_{L \in \mathcal{E}(\mathcal{A})} (1 - a(L)^2)^{l_{\mathcal{P}(\mathcal{A})}(L)} \quad .$$

Proof of Varchenko's Theorem for central arrangements.
Let $\mathcal{A} := \{H_1, \ldots, H_r\}$ be a central arrangement of hyperplanes in $\mathbb{R}^n$. We define a linear order on the hyperplanes by

$$H_1 < H_2 < \ldots < H_r \ .$$

Let $H_d \in \mathcal{A}$. Our goal is to show that M_{H_d} (for notation see Theorem 3.35) is a lower triangular block matrix and to calculate its determinant. By using Theorem 3.35 we will go back to the determinant of the Varchenko Matrix $B(\mathcal{A})$.

Determine the blocks on the diagonal of M_{H_d}:
Let $F \in \mathcal{F}(\mathcal{A})$ be a face of the arrangement $\mathcal{A}$ such that $F \subseteq H_d$. We define the set

$$P_{H_d,F}(\mathcal{A}) := \{P \in \mathcal{P}(\mathcal{A}) \mid \bar{P} \cap H_d = \bar{F}\} \ .$$

Let $k := \#P_{H_d,F}(\mathcal{A})$. Then we define a linear order on $P_{H_d,F}(\mathcal{A})$ by

$$P_1 < (P_1)_{(-F)} < P_2 < (P_2)_{(-F)} < \ldots < P_k < (P_k)_{(-F)} \tag{15}$$

where $(P_j)_{(-F)}$ is the image of P_j under the reflection in F.
Let $M_{H_d,F}$ denote the matrix, whose rows and columns are indexed by the regions in $\mathcal{P}_{H_d,F}(\mathcal{A})$ by using the linear ordering from above. (In the case that $\#\mathcal{P}_{H_d,F}(\mathcal{A}) = 0$ we get that $M_{H_d,F}$ is a 0×0-matrix.) Then Theorem 3.35 - since $M_{H_d,F}$ is a submatrix of M_{H_d} - leads to

$$(M_{H_d,F})_{i,j} = \begin{cases} 1 & \text{if } i = j \\ -\mu_{P_j}(\hat{0}, P_i) \cdot B(P_i, P_j) & \text{if } d = \max\{k \mid H_k \in T(P_i, P_j)\} \\ 0 & \text{otherwise} \end{cases}$$

By definition $\mathcal{P}_{H_d,F}(\mathcal{A}) \subseteq star(F)$ holds. For calculating the Möbius function, it is sufficient to consider $star(F)$ as a central arrangement $\mathcal{A}^F$. Then project this central arrangement onto a $n - \dim(F)$-dimensional normal to F, for obtaining an essential arrangement. We denote this projection by p.

Let $P \in \mathcal{P}_{H_d,F}(\mathcal{A})$. Then

$$p(\bar{P}) \cap p(H_d) = p(\bar{P} \cap H_d) = p(F) = \{0\}$$

which leads to $\dim(p(\bar{P}) \cap p(H_d)) = 0$.
Let Z be be a parallel hyperplane to $p(H_d)$, such that $p(H_d) \cap p(P) \neq \emptyset$. Then by Lemma 3.36 we get $\hat{P} := Z \cap p(P)$ is bounded.
Using the first part of Theorem 3.17 for calculating the Möbius function, we get

$$(M_{H_d,F})_{i,j} = \begin{cases} 1 & \text{if } i = j \\ -(-1)^{n-\dim(F)} \cdot B(P_i, P_j) & \text{if } P_j = (P_i)_{(-F)} \text{ and} \\ & d = \max\{k \mid H_k \in T(P_i, P_j)\} \\ 0 & \text{otherwise} \end{cases}$$

Thus we get by Lemma 3.38

$$(M_{H_d,F})_{i,j} = \begin{cases} 1 & \text{if } i = j \\ (-1)^{n-\dim(F)+1} \cdot a(L) & \text{if } P_j = (P_i)_{(-F)}, \\ & d = \max\{k \mid H_k \in T(P_i, P_j)\} \\ & \text{and } L \text{ denotes the continuation of } F \\ 0 & \text{otherwise} \end{cases}$$

Calculating the determinant of $M_{H_d,F}$:
Let $P, Q \in \mathcal{P}_{H_d,F}(\mathcal{A})$. Then $T(P, (P)_{(-F)}) = T(Q, (Q)_{(-F)})$ holds, because of symmetry. Therefore, if $d \neq \max\{k \mid H_k \in T(P, (P)_{(-F)})\}$ for $P \in \mathcal{P}_{H_d,F}(\mathcal{A})$, this leads to $M_{H_d,F} = \mathrm{Id}$. If $d = \max\{k \mid H_k \in T(P, (P)_{(-F)})\}$ for $P \in \mathcal{P}_{H_d,F}(\mathcal{A})$, we get:

- The entries on the main diagonal of $M_{H_d,F}$ equal 1.
- The entries on the secondary diagonals of $M_{H_d,F}$ equal $-(-1)^{n-k+1} \cdot a(L)$, where L is the continuation of F.

Define

$$e_{d,F}(L) := \begin{cases} 0 & \text{if } d \neq \max\{k \mid H_k \in T(P, (P)_{(-F)})\} \\ & \text{for } P \in \mathcal{P}_{H_d,F}(\mathcal{A}) \\ \frac{1}{2}\#\mathcal{P}_{H_d,F}(\mathcal{A}) & \text{if } d = \max\{k \mid H_k \in T(P, (P)_{(-F)})\} \\ & \text{for } P \in \mathcal{P}_{H_d,F}(\mathcal{A}) \end{cases} \tag{16}$$

Thus we get
$$\det M_{H_d,F} = (1 - a(L)^2)^{e_{d,F}(L)}$$

Show, that M_{H_d} is a lower triangular matrix:
Now we use some of the conclusions of Lemma 3.39:
Let $\mathcal{P}_{H_d,F}(\mathcal{A}), \mathcal{P}_{H_d,F'}(\mathcal{A})$ such that $F \neq F'$. Then we have to distinguish two cases:

Case 1: $\dim(F) = \dim(F')$

Let $P \in \mathcal{P}_{H_d,F}(\mathcal{A}), Q \in \mathcal{P}_{H_d,F'}(\mathcal{A})$. Then because of (14) (Lemma 3.39) holds

$$P \notin star(F') \ .$$

Since by the definition of $\mathcal{P}_{H_d,F'}(\mathcal{A})$ the face F' is the maximal face of Q which is contained in H_d, this leads by the second part of Theorem 3.17 to

$$\mu_Q(\hat{0} = Q, P) = 0 \ .$$

Analogically we get

$$\mu_P(\hat{0} = P, Q) = 0 \ .$$

Case 2: $\dim(F) < \dim(F')$

Let $P \in \mathcal{P}_{H_d,F}(\mathcal{A})$, $Q \in \mathcal{P}_{H_d,F'}(\mathcal{A})$. Then because of (13) (Lemma 3.39) holds

$$P \notin star(F') \ .$$

This leads again by the second part of Theorem 3.17 to

$$\mu_Q(\hat{0} = Q, P) = 0 \ .$$

Let $F \in \mathcal{F}(\mathcal{A})$ such that $\mathcal{P}_{H_d,F}(\mathcal{A}) \neq \emptyset$. We define a partial order on this non-empty sets $\mathcal{P}_{H_d,F}(\mathcal{A})$ by:

$$\mathcal{P}_{H_d,F}(\mathcal{A}) \leq \mathcal{P}_{H_d,F'}(\mathcal{A}) \Leftrightarrow \dim(F) \leq \dim(F')$$

We extend this partial order to a linear order and extend this again to a linear order on $\mathcal{P}(\mathcal{A})$ by using the order defined in the first part of this proof (see (15)). The extension to this linear order is well-defined because of (11) and (12) (see Lemma 3.39).

Let $F_{1,1}, \ldots, F_{1,r_1}, \ldots, F_{l,1}, \ldots F_{l,r_l} \in \mathcal{F}(\mathcal{A})$ such that $F_{i,j} \subseteq H_d$ and $\mathcal{P}_{H_d,F_{i,j}}(\mathcal{A}) \neq \emptyset$, $1 \leq i \leq l$, $1 \leq j \leq r_i$. The last condition verifies that $M_{H_d,F_{i,j}}$ is not a 0×0-Matrix. Let $\dim F_{i,j_1} = \dim F_{i,j_2}$, $1 \leq i \leq l$ and $1 \leq j_1, j_2 \leq r_i$, and $\dim F_{i_1,j_1} < \dim F_{i_2,j_2}$ if $i_1 < i_2$ for $1 \leq i_1, i_2 \leq l$, $1 \leq j_1 \leq r_{i_1}$, $1 \leq j_2, \leq r_{j_2}$. As described above we get a linear order on the sets $\mathcal{P}_{H_d,F_{i,j}}(\mathcal{A})$, $1 \leq i \leq l$, $1 \leq j \leq r_i$:

$$\mathcal{P}_{H_d,F_{1,1}}(\mathcal{A}) \leq \ldots \leq \mathcal{P}_{H_d,F_{1,r_1}}(\mathcal{A}) \leq \ldots \leq \mathcal{P}_{H_d,F_{l,1}}(\mathcal{A}) \leq \ldots \mathcal{P}_{H_d,F_{l,r_l}}(\mathcal{A}) \ .$$

We extend this to a linear order on $\mathcal{P}(\mathcal{A})$ and index the rows and columns of M_{H_d} by this linear order.

Then by the results of Case 1 and Case 2 the matrix M_{H_d} has the following block structure:

$$
M_{H_d} = \begin{array}{r l}
\mathcal{P}_{H_d,F_{1,1}}(\mathcal{A}) \{ & \\
\vdots & \\
\mathcal{P}_{H_d,F_{1,r_1}}(\mathcal{A}) \{ & \\
\vdots & \\
\mathcal{P}_{H_d,F_{l,1}}(\mathcal{A}) \{ & \\
\vdots & \\
\mathcal{P}_{H_d,F_{l,r_l}}(\mathcal{A}) \{ &
\end{array}
\left(\begin{array}{ccccccc}
\boxed{M_{H_d,F_{1,1}}} & 0 & 0 & & & & \\
0 & \ddots & 0 & 0 & & 0 & \\
0 & 0 & \boxed{M_{H_d,F_{1,r_1}}} & & & & \\
 & \star & & \ddots & & 0 & \\
 & & & & \boxed{M_{H_d,F_{l,1}}} & 0 & 0 \\
 & \star & & \star & 0 & \ddots & 0 \\
 & & & & 0 & 0 & \boxed{M_{H_d,F_{l,r_l}}}
\end{array}\right)
$$

Calculating the determinant of M_{H_d}:
Hence we can calculate the determinant of M_{H_d} by calculating the determinant of its blocks:

$$\begin{aligned}\det M_{H_d} &= \prod_{F\in\mathcal{F}(\mathcal{A}),F\subseteq H_d} \det M_{H_d,F} \\ &= \prod_{F\in\mathcal{F}(\mathcal{A}),F\subseteq H_d} (1-a(L)^2)^{e_{d,F}(L)} \quad L \text{ denotes the continuation of } F\end{aligned}$$

We define

$$e_d(L) := \sum_{F\in\mathcal{F}(\mathcal{A}),F\subseteq H_d} e_{d,F}(L)$$

Using the definition leads to

$$\det M_{H_d} = \prod_{L\in\mathcal{E}(\mathcal{A}),L\subseteq H_d} (1-a(L)^2)^{e_d(L)} \tag{17}$$

By Lemma 3.40 and the definition of $e_d(L)$ (for the definition of $e_{d,F}(L)$ see (16)) we get that $e_d(L)$ equals half the number of regions $P \in \mathcal{P}(\mathcal{A})$, such that L is the minimal intersection which contains $H_d \cap \bar{P}$, if $d = \max\{k \mid H_k \subseteq L\}$. Therefore, we have $e_d(L) = l_{\mathcal{P}(\mathcal{A})}(L)$, if $d = \max\{k \mid H_k \subseteq L\}$.
If $d \neq \max\{k \mid H_k \subseteq L\}$, then $e_d(L) = 0$ holds, since in this case $e_{d,F}(L) = 0$ by definition.

Calculating the determinant of $B(\mathcal{A})$:
Using Theorem 3.35 we get

$$\begin{aligned}\det B(\mathcal{A}) &= \det M_{H_1} \cdot \det M_{H_2} \cdot \ldots \cdot \det M_{H_r} \\ &= \prod_{H_d\in\mathcal{A}} \prod_{L\in\mathcal{E}(\mathcal{A}),L\subseteq H_d} (1-a(L)^2)^{e_d(L)} \qquad \text{see (17)} \\ &\overset{*}{=} \prod_{L\in\mathcal{E}(\mathcal{A})} (1-a(L)^2)^{l_{\mathcal{P}(\mathcal{A})}(L)}\end{aligned}$$

The last equation, which is labelled by $*$, holds because of the arguments from above concerning the exponents and because of the fact that for each $L \in \mathcal{E}(\mathcal{A})$ exists a unique hyperplane H_d such that $L \subseteq H_d$ and $d = \max\{k \mid H_k \subseteq L\}$.
Therefore, we have proved Varchenko's Theorem for central arrangements. □

3.6 Example

Next we consider a concrete example. The aim of the detailed calculations in this example is to show how the main theorems of this chapter can be applied to a concrete situation and to illustrate the proof of Theorem 3.5. A similar is mentioned but not worked out in detail in [7].

Let $\mathcal{A}$ be an essential arrangement consisting of three hyperplanes in $\mathbb{R}^2$ (see Figure 3.VIII).
For generating the matrix $B(\mathcal{A})$, we order the regions by $P_1 < P_2 < P_3 < P_4 < P_5 < P_6$ (see Figure 3.VIII). We order the hyperplanes by $H_1 < H_2 < H_3$. For the factorisation of $B(\mathcal{A})$ we calculate the matrices $M_{H_1}, M_{H_2}, M_{H_3}$. We use for each matrix M_{H_i}, $1 \leq i \leq 3$, an ordering of the regions which is given by the partition of $\mathcal{P}(\mathcal{A})$ into the sets $\mathcal{P}_{H_i,F_{ij}}(\mathcal{A})$, where F_{ij} denotes a face which is contained in H_i. We order the faces by

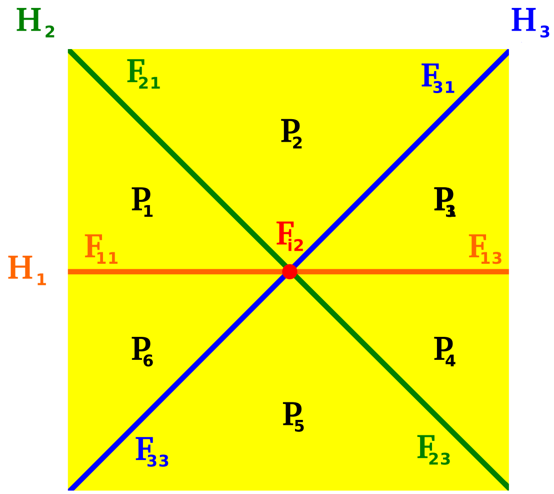
$$F_{i,2} < F_{i,1} < F_{i,3}$$

First, we calculate the submatrices $M_{H_i,F_{i,j}}$ by the formula which we have developed in the proof of Theorem 3.5 (see Figure 3.IX, Figure 3.X, Figure 3.XI). Then we use Theorem 3.35 for calculating the remaining entries of M_{H_i}, $1 \leq i \leq 3$. We know that for $P \neq Q$

$$M_{H_i}(P, Q) = 0 \text{ if } i \neq \max\{k \mid H_k \in T(P, Q)\}$$

Thus we have to calculate $M_{H_1}(P, Q)$ for $P, Q \in \{P_1, P_6\}$ and $P, Q \in \{P_3, P_4\}$. Since $\mathcal{P}_{H_1,F_{11}}(\mathcal{A}) = \{P_1, P_6\}$ and $\mathcal{P}_{H_1,F_{13}}(\mathcal{A}) = \{P_3, P_4\}$ we are already done for this matrix:

$$M_{H_1} = \begin{array}{c} \\ P_2 \\ P_5 \\ P_1 \\ P_6 \\ P_3 \\ P_4 \end{array} \!\!\! \begin{array}{c} \begin{array}{cccccc} P_2 & P_5 & P_1 & P_6 & P_3 & P_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & a_{H_1} & 0 & 0 \\ 0 & 0 & a_{H_1} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & a_{H_1} \\ 0 & 0 & 0 & 0 & a_{H_1} & 1 \end{pmatrix} \end{array}$$



For generating the matrix $B(\mathcal{A})$, we order the regions by $P_1 < P_2 < P_3 < P_4 < P_5 < P_6$:

$$B(\mathcal{A}) = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{array} \begin{array}{c} \begin{array}{cccccc} P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \end{array} \\ \begin{pmatrix} 1 & a_{H_2} & a_{H_2}a_{H_3} & a_{H_1}a_{H_2}a_{H_3} & a_{H_1}a_{H_3} & a_{H_1} \\ a_{H_2} & 1 & a_{H_3} & a_{H_1}a_{H_3} & a_{H_1}a_{H_2}a_{H_3} & a_{H_1}a_{H_2} \\ a_{H_2}a_{H_3} & a_{H_3} & 1 & a_{H_1} & a_{H_1}a_{H_2} & a_{H_1}a_{H_2}a_{H_3} \\ a_{H_1}a_{H_2}a_{H_3} & a_{H_1}a_{H_3} & a_{H_1} & 1 & a_{H_2} & a_{H_2}a_{H_3} \\ a_{H_1}a_{H_3} & a_{H_1}a_{H_2}a_{H_3} & a_{H_1}a_{H_2} & a_{H_2} & 1 & a_{H_3} \\ a_{H_1} & a_{H_1}a_{H_2} & a_{H_1}a_{H_2}a_{H_3} & a_{H_2}a_{H_3} & a_{H_3} & 1 \end{pmatrix} \end{array}$$

Figure 3.VIII: Essential arrangement $\mathcal{A} := \{H_1, H_2, H_3\}$ in $\mathbb{R}^2$

For getting M_{H_2} we have to calculate $M_{H_2}(P,Q)$ for $\{P,Q\} \in \{\{P_3,P_5\},\{P_2,P_6\}\}$. Since by Theorem 3.35 holds

$$M_{H_2}(P,Q) = -\mu_Q(\hat{0},P) \cdot B(P,Q)$$

we have to draw the Edelman Posets for the four cases from above. For this posets and the values of the Möbius function see Figure 3.XII. As shown in the proof of Theorem 3.5 we get by the ordering of the faces $F_{2,i}$ and the induced order on the sets $\mathcal{P}_{H_2,F_{2,j}}(\mathcal{A})$ a block-matrix which is lower triangular:

$$M_{H_2} = \begin{array}{c} \\ P_3 \\ P_6 \\ P_1 \\ P_2 \\ P_4 \\ P_5 \end{array} \begin{array}{c} \begin{array}{cccccc} P_3 & P_6 & P_1 & P_2 & P_4 & P_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & a_{H_2} & 0 & 0 \\ 0 & a_{H_1}a_{H_2} & a_{H_2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & a_{H_2} \\ a_{H_1}a_{H_2} & 0 & 0 & 0 & a_{H_2} & 1 \end{pmatrix} \end{array}$$

For completing the matrix M_{H_3} we have to calculate $M_{H_3}(P,Q)$ for $\{P,Q\} \in \{\{P_1,P_3\},\{P_1,P_5\},\{P_2,P_4\},\{P_2,P_5\},\{P_3,P_6\},\{P_4,P_6\}\}$ (see Figure 3.XIII). Note that again M_{H_3} is a lower triangular block matrix:

$$M_{H_3} = \begin{array}{c} \\ P_1 \\ P_4 \\ P_2 \\ P_3 \\ P_5 \\ P_6 \end{array} \begin{array}{c} \begin{array}{cccccc} P_1 & P_4 & P_2 & P_3 & P_5 & P_6 \end{array} \\ \begin{pmatrix} 1 & -a_{H_1}a_{H_2}a_{H_3} & 0 & 0 & 0 & 0 \\ -a_{H_1}a_{H_2}a_{H_3} & 1 & 0 & 0 & 0 & 0 \\ 0 & a_{H_1}a_{H_3} & 1 & a_{H_3} & 0 & 0 \\ a_{H_2}a_{H_3} & 0 & a_{H_3} & 1 & 0 & 0 \\ a_{H_1}a_{H_3} & 0 & 0 & 0 & 1 & a_{H_3} \\ 0 & a_{H_2}a_{H_3} & 0 & 0 & a_{H_3} & 1 \end{pmatrix} \end{array}$$

Let Q_{H_i} denote the permutation matrix which is needed for changing the order of the columns respectively the rows of M_{H_i} to the order of the rows and columns of $B(\mathcal{A})$. Then by Theorem 3.35 holds

$$B(\mathcal{A}) = Q_{H_1}^{-1} M_{H_1} Q_{H_1} \cdot Q_{H_2}^{-1} M_{H_2} Q_{H_2} \cdot Q_{H_3}^{-1} M_{H_3} Q_{H_3} \ .$$

Therefore, we get

$$\det B(\mathcal{A}) = \det M_{H_1} \cdot \det M_{H_2} \cdot \det M_{H_3} \ .$$

We use that the matrices M_{H_i}, $1 \leq i \leq 3$ are upper triangular block matrices.

$$\begin{aligned}
\det M_{H_1} &= \det M_{H_1,F_{1,2}} \cdot \det M_{H_1,F_{1,1}} \cdot \det M_{H_1,F_{1,3}} \\
&= 1 \cdot (1-a_{H_1}^2)^1 \cdot (1-a_{H_1}^2)^1 \\
&= (1-(a_{H_1}a_{H_2}a_{H_2})^2)^0 \cdot (1-a_{H_1}^2)^{\frac{1}{2}\#\mathcal{P}_{H_1,F_{1,1}}(\mathcal{A})} \cdot (1-a_{H_1}^2)^{\frac{1}{2}\#\mathcal{P}_{H_1,F_{1,3}}(\mathcal{A})} \\
\det M_{H_2} &= \det M_{H_2,F_{2,2}} \cdot \det M_{H_2,F_{2,1}} \cdot \det M_{H_2,F_{2,3}} \\
&= 1 \cdot (1-a_{H_2}^2)^1 \cdot (1-a_{H_2}^2)^1 \\
&= (1-(a_{H_1}a_{H_2}a_{H_2})^2)^0 \cdot (1-a_{H_2}^2)^{\frac{1}{2}\#\mathcal{P}_{H_2,F_{2,1}}(\mathcal{A})} \cdot (1-a_{H_2}^2)^{\frac{1}{2}\#\mathcal{P}_{H_2,F_{2,3}}(\mathcal{A})} \\
\det M_{H_3} &= \det M_{H_3,F_{3,2}} \cdot \det M_{H_3,F_{3,1}} \cdot \det M_{H_3,F_{3,3}} \\
&= (1-(a_{H_1}a_{H_2}a_{H_3})^2)^1 \cdot (1-a_{H_3}^2)^1 \cdot (1-a_{H_3}^2)^1 \\
&= (1-(a_{H_1}a_{H_2}a_{H_2})^2)^{\frac{1}{2}\#\mathcal{P}_{H_3,F_{3,2}}(\mathcal{A})} \cdot (1-a_{H_3}^2)^{\frac{1}{2}\#\mathcal{P}_{H_3,F_{3,1}}(\mathcal{A})} \\
&\quad \cdot (1-a_{H_3}^2)^{\frac{1}{2}\#\mathcal{P}_{H_3,F_{3,3}}(\mathcal{A})}
\end{aligned}$$

Thus

$$\begin{aligned}
\det B(\mathcal{A}) = \quad & (1-(a_{H_1}a_{H_2}a_{H_2})^2)^{\frac{1}{2}\#\mathcal{P}_{H_3,F_{3,2}}(\mathcal{A})} \\
& \cdot (1-a_{H_1}^2)^{\frac{1}{2}(\#\mathcal{P}_{H_1,F_{1,1}}(\mathcal{A})+\#\mathcal{P}_{H_1,F_{1,3}}(\mathcal{A}))} \\
& \cdot (1-a_{H_2}^2)^{\frac{1}{2}(\#\mathcal{P}_{H_2,F_{2,1}}(\mathcal{A})+\#\mathcal{P}_{H_2,F_{2,3}}(\mathcal{A}))} \\
& \cdot (1-a_{H_3}^2)^{\frac{1}{2}(\#\mathcal{P}_{H_3,F_{3,1}}(\mathcal{A})+\#\mathcal{P}_{H_3,F_{3,3}}(\mathcal{A}))}
\end{aligned}$$

Now we take the set of intersections $\mathcal{E}(\mathcal{A}) = \{\mathbb{R}^2, H_1, H_2, H_3, H_1 \cap H_2 \cap H_3\}$ of the arrangement $\mathcal{A}$ into consideration: H_i is the continuation of $F_{i,1}, F_{i,3}$ for $1 \leq i \leq 3$ and $F_{i,2}$ equals the intersection $H_1 \cap H_2 \cap H_3$. Therefore, it holds

$$\begin{aligned}
\det B(\mathcal{A}) = \quad & (1-a(H_1)^2)^{\frac{1}{2}(\#\mathcal{P}_{H_1,F_{1,1}}(\mathcal{A})+\#\mathcal{P}_{H_1,F_{1,3}}(\mathcal{A}))} \\
& \cdot (1-a(H_2)^2)^{\frac{1}{2}(\#\mathcal{P}_{H_2,F_{2,1}}(\mathcal{A})+\#\mathcal{P}_{H_2,F_{2,3}}(\mathcal{A}))} \\
& \cdot (1-a(H_3)^2)^{\frac{1}{2}(\#\mathcal{P}_{H_3,F_{3,1}}(\mathcal{A})+\#\mathcal{P}_{H_3,F_{3,3}}(\mathcal{A}))} \\
& \cdot (1-a(H_1 \cap H_2 \cap H_3)^2)^{\frac{1}{2}\#\mathcal{P}_{H_3,F_{3,2}}(\mathcal{A})}
\end{aligned}$$

By Lemma 3.40 and since $l_{\mathcal{P}(\mathcal{A})}(\mathbb{R}^2) = 0$ (see Remark 3.6) we get

$$\det B(\mathcal{A}) = \prod_{L \in \mathcal{E}(\mathcal{A})} (1-a(L)^2)^{l_{\mathcal{P}(\mathcal{A})}(L)} .$$

This equals the result which we obtain by Theorem 3.5.

H_1			
F_{11}	$\mathcal{P}_{H_1,F_{11}}(\mathcal{A}) = \{P_1, P_6\}$	$T(P_1, P_6) = \{H_1\}$ $1 = \max\{k \mid H_k \in T(P_1, P_6)\}$	$M_{H_1,F_{11}} = \begin{pmatrix} 1 & a_{H_1} \\ a_{H_1} & 1 \end{pmatrix}$
F_{12}	$\mathcal{P}_{H_1,F_{12}}(\mathcal{A}) = \{P_2, P_5\}$	$T(P_2, P_5) = \{H_1, H_2, H_3\}$ $3 = \max\{k \mid H_k \in T(P_2, P_5)\}$	$M_{H_1,F_{12}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
F_{13}	$\mathcal{P}_{H_1,F_{13}}(\mathcal{A}) = \{P_3, P_4\}$	$T(P_3, P_4) = \{H_1\}$ $1 = \max\{k \mid H_k \in T(P_3, P_4)\}$	$M_{H_1,F_{13}} = \begin{pmatrix} 1 & a_{H_1} \\ a_{H_1} & 1 \end{pmatrix}$

Figure 3.IX: Calculation of the matrices $M_{H_1,F_{1j}}$, $1 \leq j \leq 3$

H_2			
F_{21}	$\mathcal{P}_{H_2,F_{21}}(\mathcal{A}) = \{P_1, P_2\}$	$T(P_1, P_2) = \{H_2\}$ $2 = \max\{k \mid H_k \in T(P_1, P_2)\}$	$M_{H_2,F_{12}} = \begin{pmatrix} 1 & a_{H_2} \\ a_{H_2} & 1 \end{pmatrix}$
F_{22}	$\mathcal{P}_{H_2,F_{22}}(\mathcal{A}) = \{P_3, P_6\}$	$T(P_3, P_6) = \{H_1, H_2, H_3\}$ $3 = \max\{k \mid H_k \in T(P_3, P_6)\}$	$M_{H_2,F_{22}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
F_{23}	$\mathcal{P}_{H_2,F_{23}}(\mathcal{A}) = \{P_4, P_5\}$	$T(P_4, P_5) = \{H_2\}$ $2 = \max\{k \mid H_k \in T(P_4, P_5)\}$	$M_{H_2,F_{23}} = \begin{pmatrix} 1 & a_{H_2} \\ a_{H_2} & 1 \end{pmatrix}$

Figure 3.X: Calculation of the matrices $M_{H_2,F_{2j}}$, $1 \leq j \leq 3$

H_3			
F_{31}	$\mathcal{P}_{H_3,F_{31}}(\mathcal{A}) = \{P_2, P_3\}$	$T(P_2, P_3) = \{H_3\}$ $3 = \max\{k \mid H_k \in T(P_2, P_3)\}$	$M_{H_3,F_{31}} = \begin{pmatrix} 1 & a_{H_3} \\ a_{H_3} & 1 \end{pmatrix}$
F_{32}	$\mathcal{P}_{H_3,F_{32}}(\mathcal{A}) = \{P_1, P_4\}$	$T(P_1, P_4) = \{H_1, H_2, H_3\}$ $3 = \max\{k \mid H_k \in T(P_1, P_4)\}$	$M_{H_3,F_{23}} =$ $\begin{pmatrix} 1 & -a_{H_1}a_{H_2}a_{H_3} \\ -a_{H_1}a_{H_2}a_{H_3} & 1 \end{pmatrix}$
F_{33}	$\mathcal{P}_{H_3,F_{33}}(\mathcal{A}) = \{P_5, P_6\}$	$T(P_5, P_6) = \{H_3\}$ $3 = \max\{k \mid H_k \in T(P_5, P_6)\}$	$M_{H_3,F_{33}} = \begin{pmatrix} 1 & a_{H_3} \\ a_{H_3} & 1 \end{pmatrix}$

Figure 3.XI: Calculation of the matrices $M_{H_3,F_{3j}}$, $1 \leq j \leq 3$

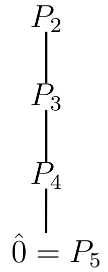
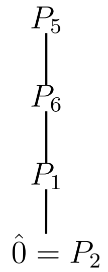
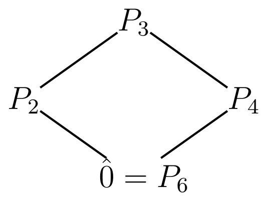
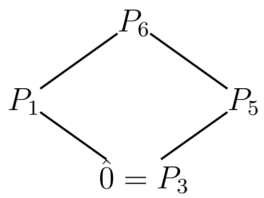
Regions	Poset	Möbius Function	Matrixentry
(P_3, P_5)		$\mu_{P_5}(\hat{0}, P_3) = 0$	0
(P_6, P_2)		$\mu_{P_2}(\hat{0}, P_6) = 0$	0
(P_2, P_6)		$\mu_{P_6}(\hat{0}, P_2) = -1$	$a_{H_1} a_{H_2}$
(P_5, P_3)		$\mu_{P_3}(\hat{0}, P_5) = -1$	$a_{H_1} a_{H_2}$

Figure 3.XII: Calculation of $M_{H_2}(P, Q)$ for P, Q such that $2 = \max\{k \mid H_k \in T(P, Q)\}$

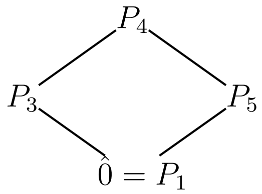
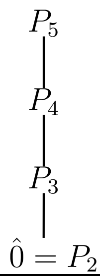
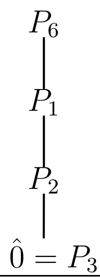
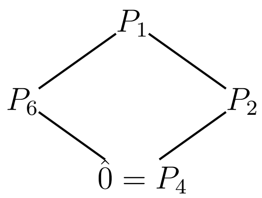
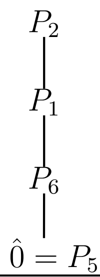
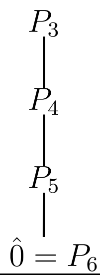
Regions	Poset	Möbius Function	Matrixentry
(P_3, P_1)		$\mu_{P_1}(\hat{0}, P_3) = -1$	$a_{H_2}a_{H_3}$
(P_5, P_1)		$\mu_{P_1}(\hat{0}, P_5) = -1$	$a_{H_1}a_{H_3}$
(P_4, P_2)		$\mu_{P_2}(\hat{0}, P_4) = 0$	0
(P_5, P_2)		$\mu_{P_2}(\hat{0}, P_5) = 0$	0
(P_1, P_3)		$\mu_{P_3}(\hat{0}, P_1) = 0$	0
(P_6, P_3)		$\mu_{P_3}(\hat{0}, P_6) = 0$	0
(P_2, P_4)		$\mu_{P_4}(\hat{0}, P_2) = -1$	$a_{H_1}a_{H_3}$
(P_6, P_4)		$\mu_{P_4}(\hat{0}, P_6) = -1$	$a_{H_2}a_{H_3}$
(P_1, P_5)		$\mu_{P_5}(\hat{0}, P_1) = 0$	0
(P_2, P_5)		$\mu_{P_5}(\hat{0}, P_2) = 0$	0
(P_4, P_6)		$\mu_{P_6}(\hat{0}, P_4)$	0
(P_3, P_6)		$\mu_{P_6}(\hat{0}, P_3) = 0$	0

Figure 3.XIII: Calculation of $M_{H_3}(P, Q)$ for P, Q such that $3 = \max\{k \mid H_k \in T(P, Q)\}$

4 Varchenko's Theorem for Cones

4.1 The bilinear form restricted to a cone

Let $\mathcal{A}$ be an arrangement of hyperplanes and $\mathcal{K}$ a cone of $\mathcal{A}$ as defined in 2.1, so we can consider $\mathcal{A}_\mathcal{K}$. To every hyperplane $H \in \mathcal{A}_\mathcal{K}$ we assign a weight a_H which we treat as an indeterminate. The weight of an intersection $L \in \mathcal{E}_\mathcal{K}(\mathcal{A})$ is denoted by $a(L)$ and defined the following way:

$$a(L) := \prod_{H : L \subseteq H} a_H$$

The ring of polynomial functions in the variables $a_H, H \in \mathcal{A}_\mathcal{K}$ with coefficients in $\mathbb{Z}$ is denoted by $\mathbb{Z}[a_H, H \in \mathcal{A}_\mathcal{K}]$. Let $M_\mathcal{K}$ be the $\mathbb{Z}[a_H, H \in \mathcal{A}_\mathcal{K}]$-module, which is freely generated by the regions of $\mathcal{P}_\mathcal{K}(\mathcal{A})$.

Definition 4.1. Let $\mathcal{A}$ be a real finite arrangement of hyperplanes and $\mathcal{K}$ a cone like above, $P_i, P_j \in \mathcal{P}_\mathcal{K}(\mathcal{A})$. Then we can define a symmetric $\mathbb{Z}[a_H, H \in \mathcal{A}_\mathcal{K}]$-bilinear form over the module $M_\mathcal{K}$ by the following:

$$B(P_i, P_j) := \prod_{H \in T_{ij}} a_H \ ,$$

with $T_{ij} := \{H \in \mathcal{A} \mid H \text{ seperates } P_i \text{ and } P_j\}$.

In the case $\mathcal{K} = \mathbb{R}^n$ the definition equals Varchenko's bilinear form, which can be found in [19] and which is used in section 3.
As the following arguments show, the bilinear form is well-defined: The cone $\mathcal{K}$ is convex. So the shortest line segment between two points contained in the cone is contained in the cone, too. Let x_i denote a point in P_i and x_j a point in P_j. Each hyperplane which separates P_i and P_j is crossed by the line-segment which connects x_i and x_j. As this line-segment is contained in the cone it crosses only hyperplanes which are contained in $\mathcal{A}_\mathcal{K}$. So the bilinear form is well-defined, because the product is only taken over weights for hyperplanes, which lie in $\mathcal{A}_\mathcal{K}$.
Remark 4.2. Since $B(\cdot, \cdot)$ is a $\mathbb{Z}[a_H, H \in \mathcal{A}_\mathcal{K}]$-bilinear form over the module $M_\mathcal{K}$,

$$\det B_\mathcal{K}(\mathcal{A}) \in \mathbb{Z}[a_H, H \in \mathcal{A}_\mathcal{K}]$$

holds.

Definition 4.3. Let $\mathcal{A}$ be an arrangement of hyperplanes, $\mathcal{K}$ a cone of $\mathcal{A}$. Let $\#\mathcal{P}_\mathcal{K}(\mathcal{A}) = r$ and fix a linear order on $\mathcal{P}_\mathcal{K}(\mathcal{A})$ by $P_1 < P_2 < ... < P_r$. Then the matrix

$$B_\mathcal{K}(\mathcal{A}) := ((B(P_i, P_j)_{1 \le i,j \le r})$$

is the Varchenko Matrix restricted to the cone $\mathcal{K}$.

If $\mathcal{K} = \mathbb{R}^n$ the matrix $B_\mathcal{K}(\mathcal{A})$ equals the usual Varchenko Matrix, which can be found in [19] and is used in section 3 for central arrangements.

4.2 The determinant of the bilinear form restricted to a cone

To each intersection $L \in \mathcal{E}_{\mathcal{K}}(\mathcal{A})$ we assign a natural number the following way:

Definition 4.4 (without restriction to the cone, see [7]). Let $\mathcal{A}$ be an arrangement, $\mathcal{K}$ a cone of $\mathcal{A}$ and $L \in \mathcal{E}_{\mathcal{K}}(\mathcal{A})$ a non-empty intersection which crosses the cone. Choose a hyperplane which contains L. Then $2 \cdot l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(L)$ equals the number of regions $P \in \mathcal{P}_{\mathcal{K}}(\mathcal{A})$, which have the property that L is the minimal (with respect to inclusion) intersection containing $\bar{P} \cap H$.

For showing that $l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(L)$ is well-defined, we show that the number of regions fulfilling the conditions in the definition is even: Let L in $\mathcal{E}_{\mathcal{K}}(\mathcal{A})$ and H a hyperplane such that $L \subseteq H$. Let $P \in \mathcal{P}_{\mathcal{K}}(\mathcal{A})$ such that L is the minimal intersection which contains $\bar{P} \cap H$. Then - since L is crossing the cone - the region which we get by reflecting P in $\bar{P} \cap H$ is contained in $\mathcal{P}_{\mathcal{K}}(\mathcal{A})$, too. Hence the number of regions $P \in \mathcal{P}_{\mathcal{K}}(\mathcal{A})$, which have the property that L is the minimal intersection containing $\bar{P} \cap H$, is even.

Our aim of this chapter is to prove the following theorem, which is the main theorem of the thesis.

Theorem 4.5 (Varchenko's Theorem for Cones). *Let $\mathcal{A}$ be an arrangement, $\mathcal{K}$ a cone of $\mathcal{A}$.*
Then for the determinant of the bilinear form restricted to the cone the following holds:

$$\det B_{\mathcal{K}}(\mathcal{A}) = \prod_{L \in \mathcal{E}_{\mathcal{K}}(\mathcal{A})} (1 - a(L)^2)^{l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(L)}$$

If $\mathcal{K} = \mathbb{R}^n$ the theorem equals Varchenko's theorem for the determinant of the bilinear form (see [19]).

4.3 Some useful properties of the determinant of the bilinear form

Lemma 4.6. *Let $\mathcal{A}$ be an arrangement of hyperplanes, $\mathcal{K}$ a cone of $\mathcal{A}$ and $B_{\mathcal{K}}(\mathcal{A})$ the matrix of the to the cone restricted bilinear form. Then the following holds: The determinant of $B_{\mathcal{K}}(\mathcal{A})$ is a polynomial in a_H^2, $H \in \mathcal{A}_{\mathcal{K}}$.*

Proof. Let $\sigma \in S_n$ be a permutation. Using the Leibniz Formula for calculating the determinant of B, σ leads to the term $s_\sigma = \pm B(P_1, P_{\sigma(1)}) \cdots B(P_n, P_{\sigma(n)})$. Let $(i_1 \cdots i_k)$ be a cycle of sigma. This cycle induces the factor

$$B(P_{i_1}, P_{i_2}) B(P_{i_2}, P_{i_3}) \cdots B(P_{i_k}, P_{i_1}) \ .$$

This factor describes a closed path, which starts and ends in P_{i_1}. Accordingly the number of crossings for each hyperplane in $H \in \mathcal{A}_\mathcal{K}$ is even. This shows, that the concerned factor is a polynomial in $a_H^2, H \in \mathcal{A}_\mathcal{K}$. As each cycle of σ induces such a factor, the assertion is proved. □

Lemma 4.7. *For all $\alpha \in \mathbb{C} \setminus \{0\}$ the polynomials $1 + \alpha x_1 \cdots x_n$ are irreducible in $\mathbb{C}[x_1, \dots, x_n]$.*

Proof. Assume, that there are $f_1, f_2 \in \mathbb{C}[x_1, \dots, x_n]$ such that $f_1 \cdot f_2 = 1 + \alpha x_1 \cdot x_n$ holds. Since the degree of x_n is additive, we can conclude without loss of generality that $\deg_{x_n}(f_1) = 1$ and $\deg_{x_n}(f_2) = 0$. This leads to $f_1 = g \cdot x_n + r_1,\ f_2 = r_2$ with $g, r_1, r_2 \in \mathbb{C}[x_1, \dots x_{n-1}]$. Hence,

$$f_1 \cdot f_2 = (g \cdot x_n + r_1) \cdot r_2 = (g \cdot r_2) \cdot x_n + r_1 \cdot r_2$$

and by assumption

$$f_1 \cdot f_2 = 1 + \alpha x_1 \cdots x_n$$

Comparing the coefficients (using $\mathbb{C}[x_1, \dots, x_n]$ is isomorphic to $\mathbb{C}[x_1, \dots, x_{n-1}][x_n]$) we get

$$1 = r_1 \cdot r_2$$

This means, that $r_1, r_2 \in \mathbb{C}^\star[x_1, \dots, x_n] = \{f \mid f \text{ is a polynomial with } \deg f = 0 \text{ and } f \neq 0\}$. Since $f_2 = r_2$ we get $f_2 \in \mathbb{C}^\star[x_1, \dots, x_n]$, in other words f_2 is a unit. Hence $1 + \alpha x_1 \dots x_n$ is irreducible in $\mathbb{C}[x_1, \dots, x_n]$. □

The following corollary is a direct consequence of Lemma 4.7:

Corollary 4.8. *Let $\mathcal{A}$ be an arrangement of hyperplanes, $\mathcal{K}$ a cone of $\mathcal{A}$ and $L \in \mathcal{E}_\mathcal{K}(\mathcal{A})$. Then $1 \pm a(L)^2$ is irreducible in $\mathbb{Z}[a_H^2, H \in \mathcal{A}_\mathcal{K}]$.*

In imitiation of writing $\mathcal{P}_\mathcal{K}(\mathcal{A})$ we define $\mathcal{P}_D(\mathcal{A}) := \{P \in \mathcal{P}(\mathcal{A}) \mid P \subseteq D\}$ for an arbitrary subarrangement $\mathcal{D} \subseteq \mathcal{A}$ and $D \in \mathcal{P}(\mathcal{D})$.
Analogue to Definition 3.4 and Definition 4.4 we define:

Definition 4.9. Let $\mathcal{A}$ be an arrangement of hyperplanes and $\mathcal{C} \subseteq \mathcal{A}$ a central subarrangement. Let $L \in \mathcal{E}(\mathcal{A})$ such that there is no $H \in \mathcal{C}$ with $L \subseteq H$ and let $D \in \mathcal{P}(\mathcal{A} \setminus \mathcal{A}^L)$. Let $H \in \mathcal{A}$ such that $L \subseteq H$. Then $2 \cdot l_{\mathcal{P}_D(\mathcal{A})}(L)$ equals the number of regions P in $\mathcal{P}_D(\mathcal{A})$ with the property that L is the minimal intersection which contains $\bar{P} \cap H$.

For the subsequent considerations recall that our regions and faces are open.

Lemma 4.10. *Let $\mathcal{A}$ be an arrangement of hyperplanes and $\mathcal{C} \subseteq \mathcal{A}$ a central subarrangement. Let $L \in \mathcal{E}(\mathcal{A})$ such that there is no $H \in \mathcal{C}$ with $L \subseteq H$ and let $D \in \mathcal{P}(\mathcal{A} \setminus \mathcal{A}^L)$. Then the following holds:*

Case 1: if $D \cap L \neq \emptyset$ then

$$l_{\mathcal{P}_D(\mathcal{A})}(L) = l_{\mathcal{P}(\mathcal{A}^L)}(L) \textit{ for the arrangement } \mathcal{A}^L.$$

Case 2: if $\bar{D} \cap L = \emptyset$ *then*

$$l_{\mathcal{P}_D(\mathcal{A})}(L) = 0 \ .$$

Case 3: if $\partial\bar{D} \cap L \neq \emptyset$ *and* $D \cap L = \emptyset$ *then*

$$l_{\mathcal{P}_D(\mathcal{A})}(L) = 0 \ .$$

Proof.
Case 1: $D \cap L \neq \emptyset$ means that L crosses the region D. Hence the set of regions which are cut by $\mathcal{A}^L$ into D is combinatorially isomorphic to the set of regions of the arrangement $\mathcal{A}^L$. Hence $l_{\mathcal{P}_D(\mathcal{A})}(L)$ equals $l_{\mathcal{P}(\mathcal{A}^L)}(L)$.
Case 2: Let H be a hyperplane such that $L \subseteq H$. Since $\bar{D} \cap L = \emptyset$, $\bar{P} \cap H$ cannot be contained in L for every region $P \subseteq D$, $P \in \mathcal{P}(\mathcal{A})$. This leads to $l_{\mathcal{P}_D(\mathcal{A})}(L) = 0$.
Case 3: L intersects D in its boundary, because $\partial\bar{D} \cap L \neq \emptyset$ and $D \cap L = \emptyset$. Let H a hyperplane such that $L \subseteq H$ and $R \subseteq D$ is a region with $\bar{R} \cap H \subseteq L$. Assume, that $\dim(\bar{R} \cap H) = \dim(L)$. Since $\bar{R} \cap H \subseteq L$, this means that there exists a face $F := \bar{R} \cap H$ of R such that $F \subseteq L$ and $\dim(F) = \dim(L)$. Since $D \cap L = \emptyset$, this face has to be contained in the boundary $\partial\bar{D}$. But this contradicts $D \in \mathcal{P}(\mathcal{A} \backslash \mathcal{A}^L)$. Hence, $\dim(\bar{R} \cap H) < \dim(L)$. Define $L' := \bigcap_{H : F \subseteq H} H$. Then $L' \in \mathcal{E}(\mathcal{A})$ and $L' \subseteq L$. By definition $F \subseteq L'$ holds. All together we get, that L is not the minimal intersection which contains $\bar{R} \cap H$. This leads to $l_{\mathcal{P}_D(\mathcal{A})}(L) = 0$. □

Remark 4.11. We assume the notation of Lemma 4.10. Then for any region D and any intersection L one of the Cases 1-3 in Lemma 4.10 occurs.

4.4 Proof of Varchenko's Theorem for Cones

For the convenience of the reader we recall Varchenko's Theorem for cones:

Theorem. *Let $\mathcal{A}$ be an arrangement, $\mathcal{K}$ a cone of $\mathcal{A}$. Then for the determinant of the bilinear form restricted to the cone the following holds:*

$$\det B_{\mathcal{K}}(\mathcal{A}) = \prod_{L \in \mathcal{E}_{\mathcal{K}}(\mathcal{A})} (1 - a(L)^2)^{l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(L)}$$

For the proof of this theorem it is sufficient to consider only central arrangements: Let $\mathcal{A} := \{H_1, \ldots, H_r\}$ be an arbitrary arrangement of hyperplanes in $\mathbb{R}^n$, $\mathcal{C} := \{H_1, \ldots, H_l\}$ a central subarrangement of $\mathcal{A}$ and $\mathcal{K} \in \mathcal{P}(\mathcal{C})$ a cone of the arrangement $\mathcal{A}$. By coning we can construct a central arrangement $\hat{\mathcal{A}} := \{\hat{H}_1, \ldots, \hat{H}_r, \hat{H}_{r+1}\}$ in $\mathbb{R}^{n+1}$. We define $\hat{\mathcal{C}} := \{\hat{H}_1, \ldots, \hat{H}_l, \hat{H}_{r+1}\}$. Then there is a region $\hat{\mathcal{K}} \in \mathcal{P}(\hat{\mathcal{C}})$ such that the set of regions which are cut into $\hat{\mathcal{K}}$ are combinatorially isomorphic to the set of regions which are cut into $\mathcal{K}$.
Hence there exists a linear ordering of the regions in $\mathcal{P}_{\mathcal{K}}(\mathcal{A})$ and $\mathcal{P}_{\hat{\mathcal{K}}}(\hat{\mathcal{A}})$ such that

$$B_{\mathcal{K}}(\mathcal{A}) = B_{\hat{\mathcal{K}}}(\hat{\mathcal{A}}) \ .$$

Therefore
$$\det B_{\mathcal{K}}(\mathcal{A}) = \det B_{\hat{\mathcal{K}}}(\hat{\mathcal{A}}) \ .$$

Using the results of the previous sections and subsections we are now in position to prove Varchenko's Theorem for Cones.

Proof of Varchenko's Theorem for Cones. Let $\mathcal{A}$ be a central arrangement of hyperplanes and $\mathcal{C} \subseteq \mathcal{A}$ a central subarrangement. The subarrangement $\mathcal{C}$ defines by its regions the cones of the arrangement $\mathcal{A}$.

Changing the weights a_H for $H \in \mathcal{C}$ to zero: Let $\mathcal{P}(\mathcal{C}) := \{\mathcal{K}_1, \ldots, \mathcal{K}_r\}$. We fix a partial order on $\mathcal{P}(\mathcal{A})$, such that

$$P_i < P_j \Leftrightarrow P_i \subseteq \mathcal{K}_k, P_j \subseteq \mathcal{K}_l \text{ with } k < l$$

We index the rows and columns of the matrix $B(\mathcal{A})$ by an arbitrary but fixed linear extension of this partial order. We change the weights a_H for $H \in \mathcal{C}$ to zero. This leads to a block structure of the matrix $B(\mathcal{A})_{[a_H=0:H\in\mathcal{C}]}$:

$$B(\mathcal{A})_{[a_H=0:H\in\mathcal{C}]} = \begin{array}{cc} \mathcal{K}_1 \left\{ \begin{array}{c} P_{1,1} \\ \vdots \\ P_{1,m_{\mathcal{K}_1}} \end{array} \right. & \\ \mathcal{K}_2 \left\{ \begin{array}{c} P_{2,1} \\ \vdots \\ P_{2,m_{\mathcal{K}_2}} \end{array} \right. & \\ \vdots \quad \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} & \\ \mathcal{K}_r \left\{ \begin{array}{c} P_{r,1} \\ \vdots \\ P_{r,m_{\mathcal{K}_r}} \end{array} \right. & \end{array} \begin{pmatrix} \boxed{B_{\mathcal{K}_1}(\mathcal{A})} & 0 & 0 & 0 \\ 0 & \boxed{B_{\mathcal{K}_2}(\mathcal{A})} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \boxed{B_{\mathcal{K}_r}(\mathcal{A})} \end{pmatrix}$$

The block structure of the matrix leads to

$$\det B(\mathcal{A})_{[a_H=0:H\in\mathcal{C}]} = \prod_{i=1}^{r} \det B_{\mathcal{K}_i}(\mathcal{A}) \ . \tag{18}$$

Because of Theorem 3.5 (Varchenko's Theorem for central arrangements) we know, that

$$\det B(\mathcal{A})_{[a_H=0:H\in\mathcal{C}]} = \prod_{L\in\mathcal{E}(\mathcal{A})} (1 - a(L)^2)^{l_{\mathcal{P}(\mathcal{A})}(L)} \ .$$

If $L \in \mathcal{E}(\mathcal{A})$ such that there is a hyperplane $H \in \mathcal{C}$ with $L \subseteq H$, then $a(L) = 0$ since the weight $a(H)$ is set to zero. Therefore the factor $(1-a(L)^2)^{l_{\mathcal{P}(\mathcal{A})}(L)}$ equals 1. We define

$$\mathcal{E}(\mathcal{A})^* := \{L \in \mathcal{E}(\mathcal{A}) \mid \text{ there is no } H \in \mathcal{C} \text{ such that } L \subseteq H\} \ .$$

Using the definition of $\mathcal{E}(\mathcal{A})^*$ we get

$$\det B(\mathcal{A})_{[a_H=0:H\in\mathcal{C}]} = \prod_{L\in\mathcal{E}(\mathcal{A})^*} (1-a(L)^2)^{l_{\mathcal{P}(\mathcal{A})}(L)} \ . \tag{19}$$

By Lemma 4.6 we know that $\det B_{\mathcal{K}}(\mathcal{A}) \in \mathbb{Z}[a_H^2, H \in \mathcal{A}]$. Since the factors $(1-a(L)^2)$ are irreducible in $\mathbb{Z}[a_H^2, H \in \mathcal{A}]$ (Corollary 4.8), hence they are prime, and since ± 1 are the units of this ring, we get

$$\det B_{\mathcal{K}_i}(\mathcal{A}) = \pm \prod_{L\in\mathcal{E}(\mathcal{A})^*} (1-a(L)^2)^{e_{\mathcal{K}_i}(L)} \text{ for } 1 \leq i \leq r \ , \tag{20}$$

where $e_{\mathcal{K}_i}(L)$ is a natural number. Thus, inserting (20) in (18) and comparing the exponents with the exponents in (19) leads to the equation

$$l_{\mathcal{P}(\mathcal{A})}(L) = \sum_{i=1}^{r} e_{\mathcal{K}_i}(L) \tag{21}$$

for every $L \in \mathcal{E}(\mathcal{A})^*$.

Considering blocks $B_{\mathcal{K}_i}(\mathcal{A})$ for $1 \leq i \leq r$:
Let $\mathcal{K} \in \mathcal{P}(\mathcal{C})$ be a cone of the arrangement $\mathcal{A}$. We analyse the block $B_{\mathcal{K}}(\mathcal{A})$. Let $L \in \mathcal{E}(\mathcal{A})$ such that there is no $H \in \mathcal{C}$ with $L \subseteq H$, namely $L \in \mathcal{E}(\mathcal{A})^*$. Since $L \in \mathcal{E}(\mathcal{A})^*$, it holds that $\mathcal{C} \subseteq \mathcal{A} \setminus \mathcal{A}^L$ and therefore $\mathcal{P}_{\mathcal{K}}(\mathcal{A} \setminus \mathcal{A}^L)$ is well-defined. Let $\mathcal{P}_{\mathcal{K}}(\mathcal{A} \setminus \mathcal{A}^L) = \{D_1, \dots, D_s\}$. We fix a partial order on the regions $\mathcal{P}_{\mathcal{K}}(\mathcal{A})$ by

$$P_i < P_j \Leftrightarrow P_i \subseteq D_k, P_j \subseteq D_l \text{ with } k < l$$

We index the rows and columns of $B_{\mathcal{K}}(\mathcal{A})$ by an arbitrary but fixed linear extension of this partial order. We change the weights a_H for $H \notin \mathcal{A}^L$ (localisation of

$\mathcal{A}$ in L) to zero. This leads to a block structure of the matrix $B_{\mathcal{K}}(\mathcal{A})_{[a_H=0:H\notin\mathcal{A}^L]}$:

$$B_{\mathcal{K}}(\mathcal{A})_{[a_H=0:H\notin\mathcal{A}^L]} = \begin{array}{r} D_1 \left\{ \begin{array}{c} P_{1,1} \\ \vdots \\ P_{1,m_{D_1}} \end{array} \right. \\ D_2 \left\{ \begin{array}{c} P_{2,1} \\ \vdots \\ P_{2,m_{D_2}} \end{array} \right. \\ \vdots \quad \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \\ D_s \left\{ \begin{array}{c} P_{s,1} \\ \vdots \\ P_{s,m_{D_s}} \end{array} \right. \end{array} \begin{pmatrix} \boxed{B_{\mathcal{K},D_1}(\mathcal{A})} & 0 & 0 & 0 \\ 0 & \boxed{B_{\mathcal{K},D_2}(\mathcal{A})} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \boxed{B_{\mathcal{K},D_s}(\mathcal{A})} \end{pmatrix}$$

Because of the block structure of the matrix we conclude

$$\det B_{\mathcal{K}}(\mathcal{A})_{[a_H=0:H\notin\mathcal{A}^L]} = \prod_{i=1}^{s} \det B_{\mathcal{K},D_i}(\mathcal{A}) \ .$$

Since $\mathcal{A}$ is a central arrangement, $\mathcal{A}\setminus\mathcal{A}^L$ is a central arrangement, too. Therefore, we can consider the regions $D \in \mathcal{P}_{\mathcal{K}}(\mathcal{A} \setminus \mathcal{A}^L)$ also as cones of the arrangement $\mathcal{A}$. Therefore, we get analogue to the beginning of this proof, that the following holds:

$$\det B_{\mathcal{K},D_i}(\mathcal{A}) = \pm \prod_{L\in\mathcal{E}(\mathcal{A})^*} (1-a(L)^2)^{e_{\mathcal{K},D}(L)} \text{ for } 1 \le i \le s \ ,$$

where $e_{\mathcal{K},D}(L)$ denotes a natural number for $L \in \mathcal{E}(\mathcal{A})^*$.
As shown above (see (20)) it holds that

$$\det B_{\mathcal{K}}(\mathcal{A}) = \pm \prod_{L\in\mathcal{E}(\mathcal{A})^*} (1-a(L)^2)^{e_{\mathcal{K}}(L)} \ . \tag{22}$$

Therefore, the factor $1-a(L)^2$ comes in $\det B_{\mathcal{K}}(\mathcal{A})$ and in $\det B_{\mathcal{K}}(\mathcal{A})_{[a_H=0:H\notin\mathcal{A}^L]}$ with the same multiplicity. In other words it holds that

$$e_{\mathcal{K}}(L) = \sum_{D\in\mathcal{P}_{\mathcal{K}}(\mathcal{A}\setminus\mathcal{A}^L)} e_{\mathcal{K},D}(L) \tag{23}$$

for every $L \in \mathcal{E}(\mathcal{A})^*$.

We determine a lower bound for the exponent $e_{\mathcal{K}}(L)$ of the factor $1 - a(L)^2$ for $L \in \mathcal{E}(\mathcal{A})^*$ in $\det B_{\mathcal{K}}(\mathcal{A})$:
Let $D \in \mathcal{P}_{\mathcal{K}}(\mathcal{A} \backslash \mathcal{A}^L)$ such that $D \cap L \neq \emptyset$, this means L crosses the region D. Then the set of regions which are cutted into D by $\mathcal{A}^L$ are combinatorically isomorphic to set of the regions of the arrangement $\mathcal{A}^L$. Since $\mathcal{A}^L$ is a central arrangement, we can use Theorem 3.5 (Varchenko's Theorem for central arrangements). Therefore,

$$e_{\mathcal{K},D}(L) = l_{\mathcal{P}(\mathcal{A}^L)}(L) \ .$$

For $D \in \mathcal{P}_{\mathcal{K}}(\mathcal{A} \backslash \mathcal{A}^L)$ we define the set $\mathcal{P}_{\mathcal{K},D} := \{P \in \mathcal{P}(\mathcal{A}) \mid P \subseteq D\}$. Since $D \subseteq \mathcal{K}$ holds by definition, the index $\mathcal{K}$ is redundant, but we use it for marking that we are working inside a cone. By Lemma 4.10 we get that $l_{\mathcal{P}(\mathcal{A}^L)} = l_{\mathcal{P}_{\mathcal{K},D}(\mathcal{A})}(L)$ and thus it holds that

$$e_{\mathcal{K},D}(L) = l_{\mathcal{P}_{\mathcal{K},D}(\mathcal{A})}(L) \ .$$

Otherwise (see (23)), it holds that

$$e_{\mathcal{K}}(L) = \sum_{D \in \mathcal{P}_{\mathcal{K}}(\mathcal{A} \backslash \mathcal{A}^L)} e_{\mathcal{K},D}(L) \ .$$

Together, this leads to the inequality

$$e_{\mathcal{K}}(L) \geq \sum_{D \in \mathcal{P}_{\mathcal{K}}(\mathcal{A} \backslash \mathcal{A}^L) : D \cap L \neq \emptyset} l_{\mathcal{P}_{\mathcal{K},D}(\mathcal{A})}(L) \ . \tag{24}$$

Going back to the full arrangement $\mathcal{A}$ for determining $e_{\mathcal{K}}(L)$ for $L \in \mathcal{E}(\mathcal{A})^*$:
As shown above (see (18)) $\det B(\mathcal{A})_{[a_H = 0 : H \in \mathcal{C}]} = \prod_{i=1}^r \det B_{\mathcal{K}_i}(\mathcal{A})$. Thus we get

$$\det B(\mathcal{A})_{[a_H=0 : H \in \mathcal{C} \text{ or } H \notin \mathcal{A}^L]} = \prod_{i=1}^r \det B_{\mathcal{K}_i}(\mathcal{A})_{[a_H = 0 : H \notin \mathcal{A}^L]} = \prod_{i=1}^r \prod_{j=1}^{s_r} \det B_{\mathcal{K}_i, D_j}(\mathcal{A}) \ .$$

By combining (21) and (24), for the exponent of the factor $(1 - a(L)^2)$ in $\det B(\mathcal{A})_{[a_H=0 : H \in \mathcal{C} \text{ or } H \notin \mathcal{A}^L]}$ it holds that

$$l_{\mathcal{P}(\mathcal{A})}(L) = \sum_{j=1}^r e_{\mathcal{K}_i}(L) \geq \sum_{j=1}^r \sum_{D \in \mathcal{P}_{\mathcal{K}_j}(\mathcal{A} \backslash \mathcal{A}^L) : D \cap L \neq \emptyset} l_{\mathcal{P}_{\mathcal{K}_j,D}(\mathcal{A})}(L) \ . \tag{25}$$

By Lemma 4.10 we know, that only the regions D such that $D \cap L \neq \emptyset$ contribute to $l_{\mathcal{P}(\mathcal{A})}(L)$. Since $\{D \in \mathcal{P}(\mathcal{A} \setminus \mathcal{A}^L) \mid D \cap L \neq \emptyset\}$ is the disjoint union of the sets $\{D \in \mathcal{P}_{cK_j}(\mathcal{A} \setminus \mathcal{A}^L) \mid D \cap L \neq \emptyset\}$, $1 \leq j \leq r$, it holds that

$$l_{\mathcal{P}(\mathcal{A})}(L) = \sum_{j=1}^r \sum_{D \in \mathcal{P}_{\mathcal{K}_j}(\mathcal{A} \backslash \mathcal{A}^L) : D \cap L \neq \emptyset} l_{\mathcal{P}_{\mathcal{K}_j,D}(\mathcal{A})}(L) \ .$$

Using the inequality (25) from above gives:

$$\sum_{j=1}^r e_{\mathcal{K}_j}(L) = \sum_{j=1}^r \sum_{D \in \mathcal{P}_{\mathcal{K}_j}(\mathcal{A} \backslash \mathcal{A}^L) : D \cap L \neq \emptyset} l_{\mathcal{P}_{\mathcal{K}_j,D}(\mathcal{A})}(L) \ ,$$

which finally leads by using (24) to

$$e_{\mathcal{K}}(L) = \sum_{D \in \mathcal{P}_{\mathcal{K}}(\mathcal{A} \setminus \mathcal{A}^L) : D \cap L \neq \emptyset} l_{\mathcal{P}_{\mathcal{K},D}(\mathcal{A})}(L) \ . \tag{26}$$

for every cone $\mathcal{K}$ of $\mathcal{A}$.

show $e_{\mathcal{K}}(L) = l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(L)$ for $\mathcal{K} \in \mathcal{P}(\mathcal{C})$, $L \in \mathcal{E}(\mathcal{A})^*$:
Fix a hyperplane $H \in \mathcal{A}$ such that $L \subseteq H$. Using the definition we get

$$l_{P_{\mathcal{K},D}(\mathcal{A})}(L) = \frac{1}{2} \#\{P \in \mathcal{P}_{\mathcal{K},D}(\mathcal{A}) \mid L \text{ is the minimal intersection which contains } H \cap \bar{P}\}$$

By the equation (26) from above we get

$$e_{\mathcal{K}}(L) = \sum_{D \in \mathcal{P}_{\mathcal{K}}(\mathcal{A} \setminus \mathcal{A}^L) : D \cap L \neq \emptyset} l_{\mathcal{P}_{\mathcal{K},D}(\mathcal{A})}(L) \ .$$

Since $l_{\mathcal{P}_{\mathcal{K},D}(\mathcal{A})}(L) = 0$ if $\bar{D} \cap L = \emptyset$ (case 2 of Lemma 4.10) and $\partial \bar{D} \cap L \neq \emptyset$ and $D \cap L = \emptyset$ (case 3 of Lemma 4.10), we are allowed sum over all $D \in \mathcal{P}_{\mathcal{K}}(\mathcal{A} \setminus \mathcal{A}^L)$:

$$\begin{aligned} e_{\mathcal{K}}(L) &= \sum_{D \in \mathcal{P}_{\mathcal{K}}(\mathcal{A} \setminus \mathcal{A}^L)} l_{\mathcal{P}_{\mathcal{K},D}(\mathcal{A})}(L) \\ &= \sum_{D \in \mathcal{P}_{\mathcal{K}}(\mathcal{A} \setminus \mathcal{A}^L)} \frac{1}{2} \#\{P \in \mathcal{P}_{\mathcal{K},D}(\mathcal{A}) \mid L \text{ is the minimal intersection} \\ &\qquad \text{which contains } H \cap \bar{P}\} \end{aligned}$$

Because $\mathcal{P}_{\mathcal{K}}(\mathcal{A})$ is the disjoint union of $\mathcal{P}_{\mathcal{K},D}(\mathcal{A})$, $D \in \mathcal{P}(\mathcal{A} \setminus \mathcal{A}^L)$, we get

$$e_{\mathcal{K}}(L) = \frac{1}{2} \#\{P \in \mathcal{P}_{\mathcal{K}}(\mathcal{A}) \mid L \text{ is the minimal intersection which contains } H \cap \bar{P}\}$$

Hence, it is shown that

$$e_{\mathcal{K}}(L) = l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(L)$$

holds for $L \in \mathcal{E}(\mathcal{A})^*$.

Determining the formula for $\det B_{\mathcal{K}}(\mathcal{A})$:
We already know (see (22))

$$\det B_{\mathcal{K}}(\mathcal{A}) = \pm \prod_{L \in \mathcal{E}(\mathcal{A})^*} (1 - a(L)^2)^{e_{\mathcal{K}}(L)} \ .$$

Since we have shown that $e_{\mathcal{K}}(L) = l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(L)$ for $L \in \mathcal{E}(\mathcal{A})^*$, we get

$$\det B_{\mathcal{K}}(\mathcal{A}) = \pm \prod_{L \in \mathcal{E}(\mathcal{A})^*} (1 - a(L)^2)^{l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(L)} \ .$$

By the definition of $\mathcal{E}(\mathcal{A})^*$ and $\mathcal{E}_{\mathcal{K}}(\mathcal{A})$ the relation $\mathcal{E}_{\mathcal{K}}(\mathcal{A}) \subseteq \mathcal{E}(\mathcal{A})^*$ holds. If L does not cross the cone - meaning $L \notin \mathcal{E}_{\mathcal{K}}(\mathcal{A})$ - we are in the second case of Lemma 4.10. Therefore it holds that $l_{\mathcal{P}_{\mathcal{K},D}(\mathcal{A})}(L) = 0$ for all $D \in \mathcal{P}_{\mathcal{K}}(\mathcal{A} \setminus \mathcal{A}^L)$, which leads to $l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(L) = 0$. This is intern leading to $(1 - a(L)^2)^{l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(L)} = 1$. Hence, it is sufficient to take the product only over the intersections L which are contained in $\mathcal{E}_{\mathcal{K}}(\mathcal{A})$:

$$\det B_{\mathcal{K}}(\mathcal{A}) = \pm \prod_{L \in \mathcal{E}_{\mathcal{K}}(\mathcal{A})} (1 - a(L)^2)^{l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(L)} .$$

Determin the sign of $\det B_{\mathcal{K}}(\mathcal{A})$: The sign of $\det B_{\mathcal{K}}(\mathcal{A})$ is by the factorisation independent on the weights $a(H)$, $H \in \mathcal{A}_{\mathcal{K}}$. Therefore we set all the weights to zero and get

$$B_{\mathcal{K}}(\mathcal{A})_{[a_H=0: H \in \mathcal{A}_{\mathcal{K}}]} = \mathrm{Id} .$$

Thus

$$\det B_{\mathcal{K}}(\mathcal{A})_{[a_H=0: H \in \mathcal{A}_{\mathcal{K}}]} = \det \mathrm{Id} = 1$$

and hence we get

$$\det B_{\mathcal{K}}(\mathcal{A}) = \prod_{L \in \mathcal{E}_{\mathcal{K}}(\mathcal{A})} (1 - a(L)^2)^{l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(L)} .$$

This completes the proof of Varchenko's Theorem for Cones. □

4.5 Example

Next we consider a concrete example. We define the arrangement

$$\mathcal{A} := \{H_1, H_2, H_3, H_4, H_5\}$$

like drawn in Figure 4.I. Let $\mathcal{C} := \{H_4, H_5\}$ and $\mathcal{K} \in \mathcal{P}(\mathcal{C})$ the light blue region. Then $\mathcal{P}_{\mathcal{K}}(\mathcal{A}) = \{P_1, P_2, P_3, P_4, P_5\}$. Notice that the situation inside the cone $\mathcal{K}$ can only be created inside a cone: By using an arrangement of three hyperplanes we can get 4 regions (hyperplanes are parallel), 6 regions (central case or two of the hyperplanes are parallel), 7 regions (no parallel hyperplanes) but never 5 regions.

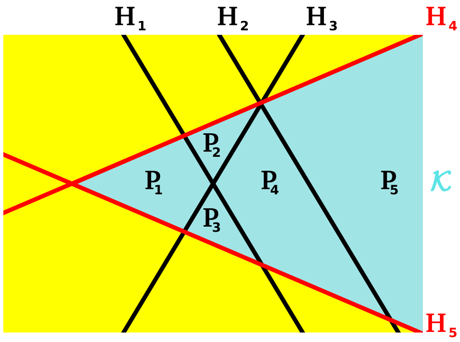


Figure 4.I: $\mathcal{A} := \{H_1, H_2, H_3, H_4, H_5\}$, $\mathcal{C} := \{H_4, H_5\}$

We order the regions by $P_1 < P_2 < P_3 < P_4 < P_5 < P_6$ and use this ordering for calculating the matrix $B_{\mathcal{K}}(\mathcal{A})$:

$$B_{\mathcal{K}}(\mathcal{A}) = \begin{array}{c c} & \begin{array}{ccccc} P_1 & P_2 & P_3 & P_4 & P_5 \end{array} \\ \begin{array}{c} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{array} & \begin{pmatrix} 1 & a_{H_1} & a_{H_3} & a_{H_1}a_{H_3} & a_{H_1}a_{H_2}a_{H_3} \\ a_{H_1} & 1 & a_{H_1}a_{H_3} & a_{H_3} & a_{H_2}a_{H_3} \\ a_{H_3} & a_{H_1}a_{H_3} & 1 & a_{H_1} & a_{H_1}a_{H_2} \\ a_{H_1}a_{H_3} & a_{H_3} & a_{H_1} & 1 & a_{H_2} \\ a_{H_1}a_{H_2}a_{H_3} & a_{H_2}a_{H_3} & a_{H_1}a_{H_2} & a_{H_2} & 1 \end{pmatrix} \end{array}$$

We calculate the determinant of $B_{\mathcal{K}}(\mathcal{A})$:

$$\det B_{\mathcal{K}}(\mathcal{A}) = (1 - a(H_1)^2)^2(1 - a(H_2)^2)^1(1 - a(H_3)^2)^2 \ .$$

Now we use Varchenko's Theorem for Cones for calculating the determinant. We have $\mathcal{E}_{\mathcal{K}}(\mathcal{A}) = \{\mathbb{R}^2, H_1, H_2, H_3, H_1 \cap H_3\}$. Hence we get by the theorem

$$\begin{aligned} \det B_{\mathcal{K}}(\mathcal{A}) =& (1 - a(\mathbb{R}^2)^2)^{l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(\mathbb{R}^2)} \cdot (1 - a(H_1)^2)^{l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(H_1)} \cdot (1 - a(H_2)^2)^{l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(H_2)} \\ & \cdot (1 - a(H_3)^2)^{l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(H_3)} \cdot (1 - a(H_1 \cap H_3)^2)^{l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(H_1 \cap H_3)} \end{aligned}$$

For determining $l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(L)$ for $L \in \mathcal{E}_{\mathcal{K}}(\mathcal{A})$, we fix a hyperplane H, which contains L. Then we count the number of regions P in $\mathcal{P}_{\mathcal{K}}(\mathcal{A})$, which have the property that L is the minimal intersection which contains $\bar{P} \cap H$. In the arrangements in Figure 4.II the corresponding regions are the dark blue ones. Thus we get

$$l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(H_1) = 2,\ l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(H_2) = 1,\ l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(H_3) = 2,\ l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(H_1 \cap H_3) = 0$$

For $\mathbb{R}^2$ holds $l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(\mathbb{R}^2) = 0$, since there is no hyperplane which contains $\mathbb{R}^2$ (see also Remark 3.6).
Together we get by Varchenko's Theorem for Cones

$$\det B_{\mathcal{K}}(\mathcal{A}) = (1-a(\mathbb{R}^2))^0 \cdot (1-a(H_1)^2)^2 \cdot (1-a(H_2)^2)^1 \cdot (1-a(H_3)^2)^2 \cdot (1-a(H_1 \cap H_3)^2)^0 \ .$$

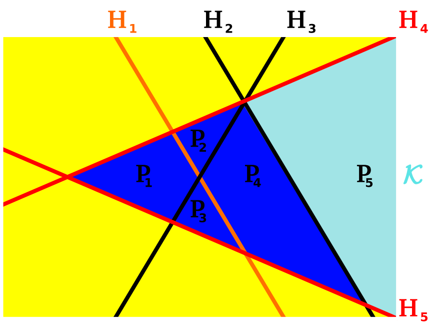
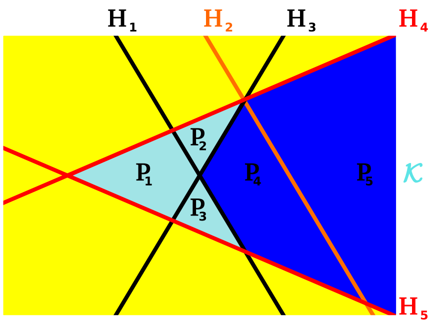
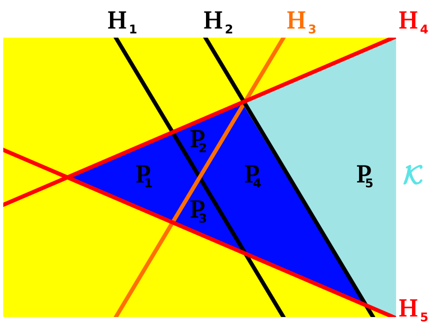
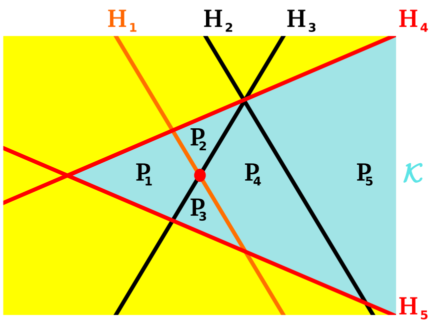
	$L = H_1$, fix H_1: $$l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(H_1) = \tfrac{1}{2}\#\{P_1, P_2, P_3, P_4\} = 2$$
	$L = H_2$, fix H_2: $$l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(H_2) = \tfrac{1}{2}\#\{P_4, P_5\} = 1$$
	$L = H_3$, fix H_3: $$l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(H_1) = \tfrac{1}{2}\#\{P_1, P_2, P_3, P_4\} = 2$$
	$L = H_1 \cap H_3$, fix H_1: $$l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(H_1) = \tfrac{1}{2}\#\{\} = 0$$

Figure 4.II: Determining $l_{\mathcal{P}_{\mathcal{K}}(\mathcal{A})}(L)$ for $L \in \mathcal{E}(\mathcal{A})$, $L \neq \mathbb{R}^2$.

5 An application of Varchenko's Theorem for Cones to Posets

We start with some basic definitions and facts which we will need throughout this chapter.
Let $P = ([n], \leq)$ denote a partially ordered set over $[n] := \{1, 2, \ldots, n\}$. The set of the linear extensions of P is denoted by $\mathcal{O}(P)$.

There is an injective map from $\mathcal{O}(P)$ to the set of permutation of $[n]$ sending the linear extension $p_1 < ... < p_n$ to the permutation σ defined by $\sigma(i) = p_i$. As usual we write $\sigma = p_1 ... p_n$ in word notation.

Let $H_{i,j} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = x_j\}$ and $\mathcal{B}_n := \{H_{i,j} \mid 1 \leq i < j \leq n\}$ the braid arrangement in $\mathbb{R}^n$. The regions of $\mathcal{B}_n$ are in bijection with the permutations in S_n via

$$R := \{(x_1, \ldots, x_n) \mid x_{\sigma(1)} < x_{\sigma(2)} < \ldots < x_{\sigma(n)}\} \leftrightarrow \sigma(1)\sigma(2)\ldots\sigma(n) := \sigma$$

Using these facts there is an obvious bijection between the linear extensions of a poset and a subset $\mathcal{P}(\mathcal{B}_n, P)$ of the set of regions $\mathcal{P}(\mathcal{B}_n)$ of $\mathcal{B}_n$. In Lemma 5.3 we show that $\mathcal{P}(\mathcal{B}_n, P)$ equals the set of regions inside a cone, which is determined by the poset P.

5.1 Posets and the Varchenko Matrix

In this section we show that it is possible to define a matrix for a poset $P = ([n], \leq)$, which we will call the Varchenko matrix of P. It will be shown that there exists a cone $\mathcal{K}$ of $\mathcal{B}_n$ such that the Varchenko Matrix for the poset P equals the Varchenko-Matrix of $\mathcal{B}_n$ restricted to $\mathcal{K}$.

Let $P = ([n], \leq)$ be a poset, $\mathcal{O}(P)$ the set of its linear extensions and $r = \#\mathcal{O}(P)$. Let σ_i, σ_j be permutations in S_n which are in bijection to linear extensions of P. Denote by R_i, R_j the regions of $\mathcal{B}_n$ which are corresponding to the permutations σ_i, σ_j. Then we define like in the previous chapters the Varchenko matrix restricted to $\mathcal{O}(P)$ by

$$B_P(\mathcal{B}_n) := (B(R_i, R_j))_{1 \leq i,j \leq r}$$

We assign to each hyperplane $H_{i,j} \in \mathcal{B}_n$ the same weight $a_{H_{i,j}} := q$.
In the following our aim is to describe the entries of $B_P(\mathcal{B}_n)$ and give a combinatorial formula for its determinant (see Theorem 5.6) under this conditions. We start with the description of the entries.
Let R_1, R_2 be regions of $\mathcal{B}_n$ corresponding to the permutations σ_1, σ_2. Then it holds that

$$B(R_1, R_2) = \prod_{H_{i,j} \text{ separates } R_1, R_2} a_{H_{i,j}} = \prod_{H_{i,j} \text{ separates } R_1, R_2} q = q^{\#\{H_{i,j} \mid H_{i,j} \text{ separates } R_1, R_2\}}.$$

In the following we denote for $\sigma \in S_n$ by $Inv(\sigma) := \{(i,j) \mid i < j \text{ and } \sigma(i) > \sigma(j)\}$ the set of inversions of σ. We are able to describe the exponent of q in a more explicit way by the following lemma:

Lemma 5.1. *Let $R_1, R_2 \in \mathcal{P}(\mathcal{B}_n)$ be regions of the braid arrangement $\mathcal{B}_n$ and σ_1, σ_2 the permutations of $[n]$, which correspond to R_1 and R_2 respectively. Then the following holds:*

$$\#\{H_{i,j} \mid H_{i,j} \textit{ separates } R_1, R_2\} = \#Inv(\sigma_2^{-1}\sigma_1)$$

Proof.

$$\begin{aligned} & H_{i,j} \text{ is separating } R_1, R_2 \\ \Leftrightarrow\ & x_i < x_j \text{ in } R_1 \text{ and } x_i > x_j \text{ in } R_2 \\ \Leftrightarrow\ & \sigma_1^{-1}(i) < \sigma_1^{-1}(j) \text{ and } \sigma_2^{-1}(i) > \sigma_2^{-1}(j) \\ \Leftrightarrow\ & \sigma_1^{-1}(i) < \sigma_1^{-1}(j) \text{ and } (\sigma_2^{-1}\sigma_1)\sigma_1^{-1}(i) > (\sigma_2^{-1}\sigma_1)\sigma_1^{-1}(j) \\ \Leftrightarrow\ & (\sigma_1^{-1}(i), \sigma_1^{-1}(j)) \text{ is an inversion of } \sigma_2^{-1}\sigma_1. \end{aligned}$$

□

Recall that a poset $P([n], \leq)$ is called naturally labelled if $1 < 2 < 3 < \ldots < n$ is one of its linear extensions.

Lemma 5.2. *Let $P := ([n], \leq)$ be a poset and $\mathcal{O}(P)$ the set of its linear extensions. Then we can find a naturally labelled poset Q on $[n]$ that is isomorphic to P as a poset such that under a suitable linear order on the linear extensions in $\mathcal{O}(P)$ and $\mathcal{O}(Q)$ holds*

$$B_P(\mathcal{B}_n) = B_Q(\mathcal{B}_n)\ .$$

Proof. Let $R_1, R_2 \in \mathcal{O}(P)$. Then the corresponding entry in the Varchenko-Matrix is defined as $B(R_1, R_2) := Inv(\sigma_2^{-1}\sigma_1)$. Changing the labelling of the poset is equivalent to multiplying σ_1, σ_2 with a permutation π by the left. Since

$$(\pi\sigma_2)^{-1} \cdot \pi\sigma_1 = \sigma_2^{-1} \cdot \pi^{-1}\pi \cdot \sigma_1 = \sigma_2^{-1}\sigma_1$$

holds, the proposition is proved. □

As a consequence we restrict ourselves to naturally labelled posets.

Since our aim is to apply Theorem 4.5 for the calculation of $\det B_P(\mathcal{B}_n)$, we have to show that there exists a subarrangement $\mathcal{C} \subseteq \mathcal{B}_n$ such that there is a suitable cone $\mathcal{K} \in \mathcal{P}(\mathcal{C})$:

Lemma 5.3. *Let $P := ([n], \leq)$ be a poset and $\mathcal{P}(\mathcal{B}_n, P)$ be the set of regions of the braid arrangement $\mathcal{B}_n$ which are in bijection to $\mathcal{O}(P)$. Then there exists a cone $\mathcal{K}$ of $\mathcal{B}_n$, such that*

$$\mathcal{P}_{\mathcal{K}}(\mathcal{B}_n) = \mathcal{P}(\mathcal{B}_n, P)\ .$$

Proof. Changing the labelling of the poset P is done by multiplying with a permutation by the left (see Lemma 5.2). This permutation corresponds to an orthogonal map in $\mathbb{R}^n$. Since orthogonal maps are mapping cones onto cones, we can assume that P is a naturally labelled poset.
Let $O := p_1 \leq \ldots \leq p_n$ be a linear extension of P. Then O corresponds to the region $R = \{(x_1, \ldots, x_n) \mid x_{p_1} < \ldots < x_{p_n}\}$. Define for each $H_{i,j} \in \mathcal{B}_n$

$$H_{i,j}^+ := \{(x_1, \ldots, x_n) \mid x_i < x_j\}$$
$$H_{i,j}^- := \{(x_1, \ldots, x_n) \mid x_i > x_j\} \ .$$

Then all regions with the property $x_i < x_j$ are contained in $H_{i,j}^+$.
Let $M := \{(i,j) \mid i < j, i, j \in P\}$ be the set of relations of the naturally labelled poset P. Then

$$\mathcal{P}(\mathcal{B}_n, P) = \{R \in \mathcal{P}(\mathcal{B}_n) \mid \text{ for all } (i,j) \in M \text{ we have } x_i < x_j \text{ for all } (x_1, \ldots, x_n) \in R\} \ .$$

This is equivalent to

$$\mathcal{P}(\mathcal{B}_n, P) = \{R \in \mathcal{P}(\mathcal{B}_n) \mid R \subseteq \bigcap_{(i,j) \in M} H_{i,j}^+\}$$

and we can define $\mathcal{K} := \bigcap_{(i,j)\in M} H_{i,j}^+$, this means $\mathcal{K}$ is a region of the subarrangement $\mathcal{C} := \{H_{i,j} \mid (i,j) \in M\}$. □

We summarize the results of this subsection in the following corollary:

Corollary 5.4. *Let $R_1, \ldots, R_r$ be a numbering on the regions in $\mathcal{P}_{\mathcal{K}}(\mathcal{B}_n)$ and let σ_i be the permutation corresponding to the region R_i. Let $\mathcal{K}$ be the cone of $\mathcal{B}_n$ such that $\mathcal{P}_{\mathcal{K}}(\mathcal{B}_n)$ is in bijection to $\mathcal{O}(P)$. Let $B_{\mathcal{K},P}(\mathcal{B}_n)$ denote the matrix of the bilinear form restricted to the cone $\mathcal{K}$. Then*

$$B_{\mathcal{K},P}(\mathcal{B}_n) = B_P(\mathcal{B}_n)$$

and for the entries of the matrix holds

$$B_{\mathcal{K},P}(\mathcal{B}_n)_{i,j} = q^{\#Inv(\sigma_j^{-1} \cdot \sigma_i)} \ .$$

5.2 Application of Varchenko's Theorem for cones

Because of the results of the previous subsection, we know that we can use Theorem 4.5 for calculating the determinant of $B_P(\mathcal{B}_n)$ (respectively $B_{\mathcal{K},P}(\mathcal{B}_n)$, if we want to focus on the cone). The aim of this chapter is to give a combinatorial description of the determinant of this matrix. We start with a definition, which will be needed for the formula of the determinant in Theorem 5.6.

Let A be an antichain in the poset $P = (X, <_p)$. Then we define the following poset:

Definition 5.5. Let $a \in A$ be arbitrary but fixed. We define the new poset $P_A = (X \setminus \{A\} \cup \{a\}, <_{P_A})$ by the following relations:

$$\begin{array}{ll}
i <_P j & \Rightarrow i <_{P_A} j \\
i <_P j \text{ for some } j \in A & \Rightarrow i <_{P_A} a \\
i <_P j \text{ for some } i \in A & \Rightarrow a <_{P_A} j \\
i <_P k \text{ for some } k \in A \text{ and } l <_P j \text{ for some } l \in A & \Rightarrow i <_{P_A} j
\end{array}$$

Because A is an antichain in P, P_A is still a poset. By construction P_A is a poset on $n - (\#A - 1)$ elements. In the sequel we will assume that P_A is labelled by $[n - (\#A - 1)]$.

The following theorem is the main result of this chapter:

Theorem 5.6. *Let $P := ([n], \leq)$ be a poset on $[n]$. Let $\mathcal{O}(P)$ be the set of linear extensions of P and B_P the Varchenko-Matrix. Then the following holds*

$$\det B_{\mathcal{K},P}(\mathcal{B}_n) = \prod_{A \text{ antichain in } P} (1 - q^{\#A \cdot (\#A-1)})^{(\#A-2)! \cdot \#\mathcal{O}(P_A)}$$

For getting in position to prove Theorem 5.6, we have to show a few lemmata in advance.

Let Π_n denote the set of all set-partitions Λ of $[n]$.

Lemma 5.7. *There exists a bijection between Π_n and $\mathcal{E}(\mathcal{B}_n)$:*

$$\varphi : \begin{array}{ccc} \Pi_n & \rightarrow & \mathcal{E}(\mathcal{B}_n) \\ \Lambda := \Lambda_1 \mid \ldots \mid \Lambda_r & \mapsto & \bigcap_{k=1}^r \left(\bigcap_{i,j \in \Lambda_k} H_{i,j} \right) := L \end{array}$$

with convention $H_{i,i} = \mathbb{R}^n$.

See also [13] for this well-known correspondence.

Proof. We show that φ is injective. Let $\Lambda := \Lambda_1 \mid \ldots \mid \Lambda_r \neq \Lambda'_1 \mid \ldots \mid \Lambda'_s := \Lambda'$. Then there exists a pair $i, j \in [n]$ such that without loss of generality $i, j \in \Lambda_k$ for some k, but there does not exist any s such that $i, j \in \Lambda'_s$. Therefore $H_{i,j} \supseteq \varphi(\Lambda)$ but $H_{i,j} \not\supseteq \varphi(\Lambda')$. Hence $\varphi(\Lambda) \neq \varphi(\Lambda')$ and φ is injective.
We show that φ is surjective. Let $L \in \mathcal{E}(\mathcal{B}_n)$. We define the relation

$$i \sim_L j \Leftrightarrow H_{i,j} \supseteq L \ .$$

This defines an equivalence relation on $[n]$:

- Since $H_{i,i} = \mathbb{R}^n$ and $\mathbb{R}^n \subseteq L \Rightarrow i \sim_L i$.

- Since $H_{i,j} = H_{j,i} \Rightarrow [i \sim_L j \Leftrightarrow j \sim_L i]$.
- Let $H_{i,j} \supseteq L$ and $H_{j,k} \supseteq L$, then $H_{i,j} \cap H_{j,k} \supseteq L \Rightarrow [i \sim_E j$ and $j \sim_L k \Rightarrow i \sim_L k]$

Then each equivalence class defines a block of a partition $\Lambda_L \in \Pi_n$. Since $\varphi(\Lambda_L) = L$ we get that φ is surjective. □

We equip $\mathcal{E}(\mathcal{B}_n)$ with a poset structure by ordering the intersections by reverse inclusion

$$L \leq_{\mathcal{E}(\mathcal{B}_n)} L' \Leftrightarrow L' \subseteq L \ .$$

As usual we consider Π_n as a poset ordered by refinement, i.e.

$$\Lambda \leq_{\Pi_n} \Gamma \Leftrightarrow \text{ the blocks of } \Gamma \text{ are unions of the blocks of } \Lambda$$

Then φ is a poset homomorphism.
The following lemma is first step for determining the exponents of the factors in Varchenko's formula for the determinant (see Theorem 4.5).

Lemma 5.8. *Let $L \in \mathcal{E}_{\mathcal{K}}(\mathcal{B}_n)$ such that $\varphi^{-1}(L) =: \Lambda_1 \mid \ldots \mid\Lambda_r$ contains more than one non-trivial block. Then $l_{\mathcal{K}}(L) = 0$.*

This result can be found without proof in [10] and [7]. For the sake of completeness, we will give a proof here.

Proof. Let R be a region, which is defined by $R := \{(x_1, \ldots, x_n) \mid x_{p_1} < \ldots < x_{p_n}\}$. Then its closure is $\bar{R} := \{(x_1, \ldots, x_n) \mid x_{p_1} \leq \ldots \leq x_{p_n}\}$.
Let $H_{l,k} \in \mathcal{B}_n$ be a hyperplane and $p_i = l, p_j = k$. Then

$$\begin{aligned} H_{l,k} \cap \bar{R} &= \{(x_1, \ldots, x_n) \mid x_{p_1} \leq x_{p_i} = \ldots = x_{p_j} \leq \ldots \leq x_{p_n}\} \text{ if } p_i < p_j \\ H_{l,k} \cap \bar{R} &= \{(x_1, \ldots, x_n) \mid x_{p_1} \leq x_{p_j} = \ldots = x_{p_i} \leq \ldots \leq x_{p_n}\} \text{ if } p_i > p_j \end{aligned}$$

For every $L \in \mathcal{E}_{\mathcal{K}}(\mathcal{B}_n)$ with $\varphi^{-1}(L) =: \Lambda_1 \mid \ldots \mid \Lambda_r$ it holds that

$$\operatorname{codim}(L) = \sum_{\Lambda_i \text{ non-trivial}} (\#\Lambda_i - 1) \ .$$

Let $L \in \mathcal{E}_{\mathcal{K}}(\mathcal{B}_n)$ for which $\varphi^{-1}(L) =: \Lambda_1 \mid \ldots \mid\Lambda_r$ contains more than one non-trivial block. We fix a hyperplane $H_{l,k}$ satisfying $L \subseteq H_{l,k}$. Hence, there exists a block Λ_i, such that $l, k \in \Lambda_i$. Then

$$\max \operatorname{codim}(H_{l,k} \cap \bar{R}) = \Lambda_i - 1 < \operatorname{codim}(L) \ ,$$

since L is an intersection, such that $\varphi^{-1}(L)$ contains more than one non-trivial block. Therefore $H_{l,k} \cap \bar{R} \not\subset L$. In other words, there do not exist any regions R, which have the property that L is the minimal intersection which contains $H_{l,k} \cap \bar{R}$. This shows $l_{\mathcal{K}}(L) = 0$, if $\varphi^{-1}(L) =: \Lambda_1 \mid \ldots \mid\Lambda_r$ contains more than one non-trivial block □

As a consequence of this lemma, only the intersections $L \in \mathcal{E}_{\mathcal{K}}(\mathcal{B}_n)$ such that $\varphi^{-1}(L)$ contains exactly one non-trivial block, can contribute a non-trivial factor to the determinant. We will express the factor $(1 - a(L)^2)$ by using a correspondence between such an intersection L and an antichain A of the poset P. We start with the required bijection.

Let Π_n^1 denote the set of partitions of $[n]$ with exactly one non-singleton (i.e. non-trivial) block and define $\mathcal{E}^1_{\mathcal{K}}(\mathcal{B}_n) := \{L \in \mathcal{E}_{\mathcal{K}}(\mathcal{B}_n) \mid \varphi^{-1}(L) \in \Pi_n^1\}$. Let $\mathcal{C}(P)$ denote the set of antichains of the poset P.

Lemma 5.9. *There is a bijection*

$$\Psi : \begin{array}{lcl} \mathcal{E}^1_{\mathcal{K}}(\mathcal{B}_n) & \to & \mathcal{C}(P) \\ L := \bigcap_{H_{i,j} \in \mathcal{D} \subseteq \mathcal{B}_n} H_{i,j} & \mapsto & A := \{l \mid H_{i,l} \in \mathcal{D} \textit{ or } H_{l,j} \in \mathcal{D}\} \end{array}$$

Proof. Let $L := \bigcap_{H_{i,j} \in \mathcal{D} \subseteq \mathcal{B}_n} H_{i,j} \in \mathcal{E}^1_{\mathcal{K}}(\mathcal{B}_n)$. In particular L crosses the cone. Since a hyperplane $H_{i,j}$ crosses the cone $\mathcal{K}$ if and only if i and j are incomparable in P, we get that $\Psi(L) = \{l \mid H_{i,l} \in \mathcal{D} \text{ or } H_{l,j} \in \mathcal{D}\}$ is an antichain in P. Hence the map Ψ is well-defined.
Injectivity: Let $L = \bigcap_{H_{i,j} \in \mathcal{D} \subseteq \mathcal{B}_n} H_{i,j}, \tilde{L} = \bigcap_{H_{i,j} \in \tilde{\mathcal{D}} \subseteq \mathcal{B}_n} H_{i,j} \in \mathcal{E}^1_{\mathcal{K}}(\mathcal{B}_n)$ be such that $L \neq \tilde{L}$. Then $\mathcal{D} \neq \tilde{\mathcal{D}}$ which by definition of Ψ directly leads to $\Psi(L) \neq \Psi(L')$. Hence Ψ is injective.
Surjectivity: Let $A \in \mathcal{C}(P)$ be an antichain. Since $\mathcal{B}_n$ is a central arrangement the intersection $L := \bigcap_{i,j \in A} H_{i,j}$ is not empty. As each hyperplane $H_{i,j} \in \mathcal{B}_n$ with $i, j \in A$ crosses the cone $\mathcal{K}$, we get that L crosses the cone, hence $L \in \mathcal{E}(\mathcal{B}_n)$. $\varphi^{-1}(L)$ is a partition of $[n]$ with exactly one non-singleton block and it follows that $L \in \mathcal{E}^1_{\mathcal{K}}(\mathcal{B}_n)$.

□

The following lemma shows the structure of the non-trivial factors of the determinant:

Lemma 5.10. *Let $L \in \mathcal{E}^1_{\mathcal{K}}(\mathcal{B}_n)$ and $\Psi(L) := A \in \mathcal{C}(P)$ and assign to each hyperplane $H_{i,j} \in \mathcal{B}_{n\mathcal{K}}$ the weight $a_{H_{i,j}} = q$. Then*

$$1 - a(L)^2 = 1 - q^{\#A \cdot (\#A - 1)}$$

Proof. We prove the lemma by a direct calculation:

$$\begin{aligned} 1 - a(L)^2 &= 1 - (\prod_{H_{i,j}: L \subseteq H_{i,j}} a_{H_{i,j}})^2 \\ &= 1 - (\prod_{H_{i,j}: L \subseteq H_{i,j}} q)^2 \\ &= 1 - (\prod_{H_{i,j}: i <_{\mathbb{N}} j \in A} q)^2 \\ &= 1 - q^{2 \cdot \#\{i,j \mid i,j \in A\}} \\ &= 1 - q^{2 \cdot \binom{\#A}{2}} \\ &= 1 - q^{\#A \cdot (\#A - 1)} \end{aligned}$$

□

We have to determine the exponent $l_{\mathcal{K}}(L)$ for $L \in \mathcal{E}^1_{\mathcal{K}}(\mathcal{B}_n)$. Therefore, we characterize the regions $R \in \mathcal{P}_{\mathcal{K}}(\mathcal{B}_n)$ having the property that L is the minimal intersection containing $\bar{R} \cap H_{l,k}$ for some $H_{l,k} \in \mathcal{B}_n$ with $L \subseteq H_{l,k}$. For the definition of $l_{\mathcal{K}}(L)$ see Definition 4.4.

Lemma 5.11. *Let $L \in \mathcal{E}^1_{\mathcal{K}}(\mathcal{B}_n)$, $H_{l,k} \supseteq L$ and $R \in \mathcal{P}_{\mathcal{K}}(\mathcal{B}_n)$ with $R := \{(x_1, \ldots, x_n) \mid x_{p_1} < \ldots < x_{p_n}\}$. Then L is the minimal intersection which contains $H_{l,k} \cap \bar{R}$ if and only if $|j - i| =$ codim (L), where i, j are the indices belonging to $p_i = l$ and $p_j = k$.*

Proof. Let R be a region, which is defined by $R := \{(x_1, \ldots, x_n) \mid x_{p_1} < \ldots < x_{p_n}\}$. Let $H_{l,k} \in \mathcal{B}_n$ be a hyperplane such that $H_{l,k} \supseteq L$ and $p_i = l, p_j = k$. Then

$$\begin{aligned} H_{l,k} \cap \bar{R} &= \{(x_1, \ldots, x_n) | \ x_{p_1} \leq \ldots \leq x_{p_i} = \ldots = x_{p_j} \leq \ldots \leq x_{p_n}\} \text{ if } p_i < p_j \\ H_{l,k} \cap \bar{R} &= \{(x_1, \ldots, x_n) \mid x_{p_1} \leq \ldots \leq x_{p_j} = \ldots = x_{p_i} \leq \ldots \leq x_{p_n}\} \text{ if } p_i > p_j \end{aligned}$$

This shows that codim $H_{l,k} \cap \bar{R} = |j - i|$ holds.
The rest follows directly from the definition of a minimal intersection
(see Definition 3.2).

□

We use the characterisation for counting the regions which contribute to $l_{\mathcal{K}}(L)$. We are able to express this number by the cardinality of the antichain A, which is in bijection to the intersection L, and by the number of linear extensions of the poset P_A (see Definition 5.5):

Lemma 5.12. *Let $L \in \mathcal{E}^1_{\mathcal{K}}(\mathcal{B}_n)$ and $A := \Psi(L) \in \mathcal{C}(P)$. Then*

$$l_{\mathcal{P}_{\mathcal{K}}(\mathcal{B}_n)}(L) = (\#A - 2)! \cdot \#\mathcal{O}(P_A) \ .$$

Proof. Let us calculate $l_{\mathcal{K}}(L)$. Let A be the antichain that is the image of L under the bijection Ψ from Lemma 5.9 and let $l, k \in A$. By definition of Ψ we have $H_{l,k} \supseteq L$. We fix $H_{l,k}$ and count how many ways for constructing a region $R \in \mathcal{P}_{\mathcal{K}}(\mathcal{B}_n)$, such that L is the minimal intersection which contains $\bar{R} \cap H_{l,k}$, exist.
First, we show that the antichain gives us $2 \cdot (\#A - 2)!$ possibilities for arranging its elements: Let $R := \{(x_1, \ldots, x_n) \mid x_{p_1} < \ldots < x_{p_n}\}$ and $p_i = l, p_j = k$. Then Lemma 5.11 tells us that $|j - i| = \operatorname{codim} L$ holds. Since $\operatorname{codim} L = \#A - 1$, we get that all the other elements of the antichain besides l, k have to be placed between $p_i = l$ and $p_j = k$. Hence we have two possibilities for arranging l and k and $(\#A - 2)!$ possibilities for arranging the elements in $A \setminus \{l, k\}$.
Second, we need the number of possibilities for arranging the antichain inside the poset P. This equals the number of linear extensions of $\mathcal{P}_A$, short $\#\mathcal{O}(P_A)$. Together we get

$$l_{\mathcal{P}_{\mathcal{K}}(\mathcal{B}_n)}(L) = \frac{1}{2} \cdot 2 \cdot (\#A - 2)! \cdot \#\mathcal{O}(P_A) = (\#A - 2)! \cdot \#\mathcal{O}(P_A)$$

□

Now we are in position to prove Theorem 5.6. First, we recall Theorem 4.5 for the convenience of the reader:

Theorem (Varchenko's Theorem for Cones)**.** *Let $\mathcal{A}$ be an arrangement of hyperplanes, a_H for $H \in \mathcal{A}$ the corresponding weights, $\mathcal{K}$ a cone of $\mathcal{A}$ and $B_{\mathcal{K}}(\mathcal{A})$ the matrix of the bilinear form. Then*

$$\det B_{\mathcal{K}}(\mathcal{A}) = \prod_{L \in \mathcal{E}_{\mathcal{K}}(\mathcal{A})} (1 - a(L)^2)^{l_{\mathcal{P}_{\mathcal{K}}(\mathcal{B}_n)}(L)} \ .$$

Proof of Theorem 5.6.

$$\begin{aligned} \det B_{\mathcal{K},P}(\mathcal{B}_n) &= \prod_{L \in \mathcal{E}_{\mathcal{K}}(\mathcal{B}_n)} (1 - a(L)^2)^{l_{\mathcal{P}_{\mathcal{K}}(\mathcal{B}_n)}(L)} && \text{see Theorem 4.5} \\ &= \prod_{A \in \mathcal{C}(P)} (1 - a(\Psi^{-1}(A))^2)^{l_{\mathcal{K}}(\Psi^{-1}(A))} && \text{see Lemma 5.9} \\ &= \prod_{A \in \mathcal{C}(P)} (1 - q^{\#A \cdot (\#A-1)})^{l_{\mathcal{K}}(\Psi^{-1}(A))} && \text{see Lemma 5.10} \\ &= \prod_{A \in \mathcal{C}(P)} (1 - q^{\#A \cdot (\#A-1)})^{(\#A-2)! \cdot \#\mathcal{O}(P_A)} && \text{see Lemma 5.12} \end{aligned}$$

Hence Theorem 5.6 is shown. □

5.3 Example

The arrangement $\mathcal{B}_4$ is projected onto a sphere in $\mathbb{R}^3$ for visualizing it. In Figure 5.II only the one side of the sphere is visble. The linear extensions of the poset P are lying in a cone defined by $\mathcal{C} := \{H_{1,2}, H_{1,4}\}$. This cone is the red marked region of $\mathcal{C}$.

The poset contains four antichains:

$$A_1 := \{1,3\}, A_2 := \{2,3\}, A_3 := \{4,3\}, A_4 := \{2,3,4\} \ .$$

We construct the posets P_{A_i} for $1 \leq i \leq 4$ (see Figure 5.I). We are interested in the number of linear extensions of this posets. Since the number of linear extensions of a poset does not depend on the labelling of the poset, we choose a natural labelling.

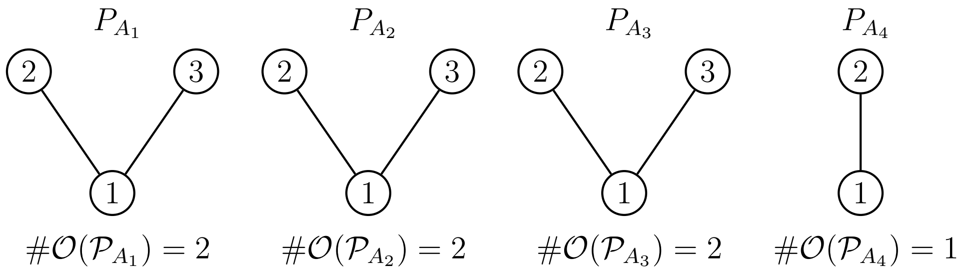


Figure 5.I: The posets P_{A_i} for $1 \leq i \leq 4$ and their number of linear extensions.

Using Theorem 5.6 we get

$$\begin{aligned}
\det B_{\mathcal{K}}(\mathcal{B}_4) &= \prod_{i=1}^{4} (1 - q^{\#A_i \cdot (\#A_i - 1)})^{(\#A_i - 2)! \cdot \#\mathcal{O}(P_{A_i})} \\
&= (1 - q^{2\cdot 1})^{0! \cdot 2} \cdot (1 - q^{2\cdot 1})^{0! \cdot 2} (1 - q^{2\cdot 1})^{0! \cdot 2} \cdot (1 - q^{3\cdot 2})^{1! \cdot 1} \\
&= (1 - q^2)^6 \cdot (1 - q^6)
\end{aligned}$$

By Theorem 4.5 we get:

$$\begin{aligned}
\det B_{\mathcal{K}}(\mathcal{B}_4) = &(1 - a(H_{1,3})^2)^2 \cdot (1 - a(H_{2,3})^2)^2 \\
&\cdot (1 - a(H_{4,3})^2)^2 \cdot (1 - a(H_{2,3} \cap H_{2,4} \cap H_{3,4})^2)^1
\end{aligned}$$

Equalizing the weights to q gives

$$\begin{aligned}
\det B_{\mathcal{K}}(\mathcal{B}_4) &= (1 - q^2)^2 \cdot (1 - q^2)^2 \cdot (1 - q^2)^2 \cdot (1 - q^{3\cdot 2})^1 \\
&= (1 - q^2)^6 \cdot (1 - q^6) \ ,
\end{aligned}$$

which equals the result from above.
Note, that for each antichain A the term $(1 - a(\bigcap_{i,j \in A} H_{i,j})^2)$ arises.

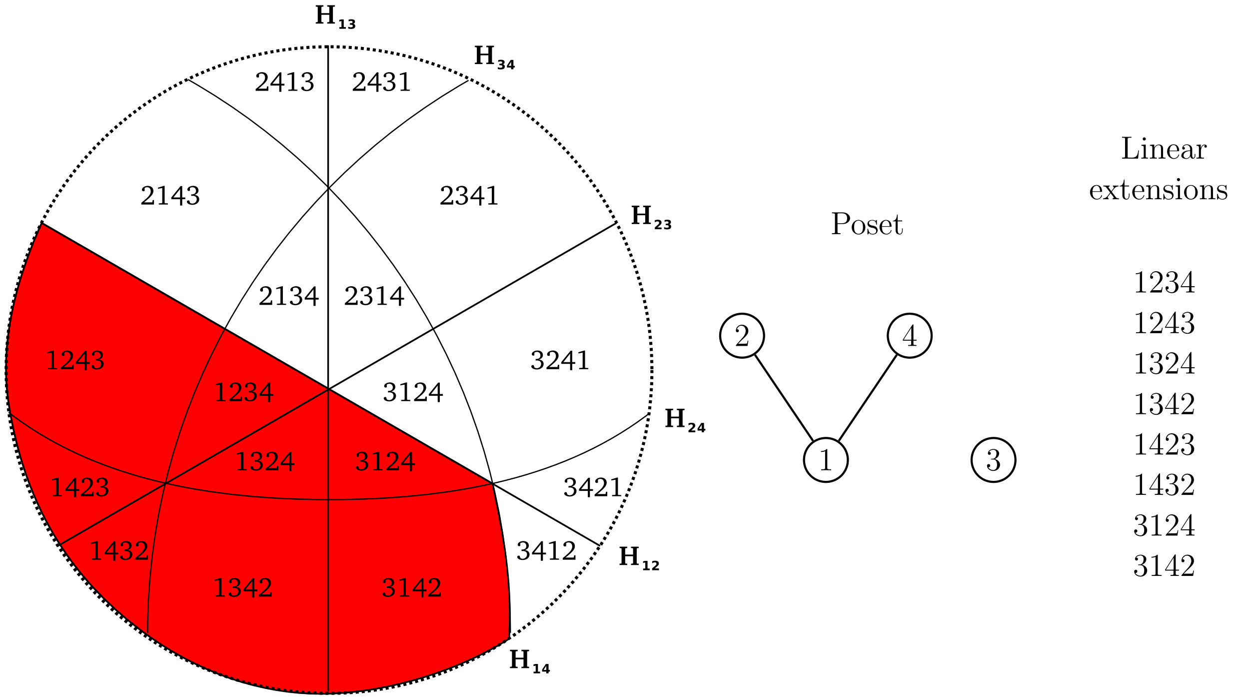


Figure 5.II: The poset (in the middle), the corresponding cone (red marked on the left) and the linear extensions (on the right)

Acknowledgements

I would like to thank several people for supporting and helping me during the development of this thesis. Thanks to …

… Volkmar Welker for his guidance throughout this thesis and for helpful and inspiring discussions.

… Graham Denham for explaining some of the ideas of his unpublished proof of Theorem 3.13 and some of the ideas of the proof of the first part of Theorem 3.17 to me.

… my parents Irmgard and Michael Gente and my brother Ralf Gente for their support and for encouraging me during the last four years.

… the other PhD-students and scientific assistants of the department for creating a pleasant working atmosphere.

References

[1] M. Aguiar, S. Mahajan: Coxeter Groups and Hopf Algebras, AMS , Providence, 2006

[2] M. Aigner: Combinatorial Theory, Springer, New York, 1979

[3] A. Björner: Topological Methods, Handbook of Combinatorics, R. Graham, M. Grötschel and L. Lovász, (Eds), North-Holland, Amsterdam, 1995, 1819-1872

[4] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, G. M. Ziegler: Oriented Matroids, Second Edition, Cambridge University Press, Cambridge, 1999

[5] G. Brightwell, P. Winkler: Counting Linear Extensions, Order 8, 1991

[6] H. Bruggeser, P. Mani: Shellable Decompositions of Cells and Spheres, Math. Scand. 29 (1971), 197-205

[7] G. Denham, P. Hanlon: Some Algebraic Properties of the Schechtman-Varchenko Bilinear Forms, New Perspectives in Geometric Combinatorics, L. J. Billera etal. eds., MSRI Publications 38 1999, 149-175

[8] A.W.M. Dress, R. Scharlau: Gated sets in metric spaces, Aequationes Math. 34 (1987), 112-120

[9] P. H. Edelman: A Partial Order on the Regions of $\mathbb{R}^n$ dissected by hyperplanes, Trans. Am. Math. Soc., Volume 283, (1984), 617-631

[10] P. Hanlon, R. P. Stanley: A q-deformation of a trivial symmetric group action, Trans. Am. Math. Soc. 350, 1998

[11] J. R. Munkres: Elements of Algebraic Topology, Perseus Books Publishing, Reading, Massachusetts, 1984

[12] J. Neggers, H. S. Kim: Basic Posets, World Scientific Publishing Co. Pte. Ltd., Singapore, 1998

[13] P. Orlik, H. Terao: Arrangements of Hyperplanes, Springer, Berlin, 1992

[14] D. Quillen: Homotopy properties of the poset of non-trivial p-subgroups of a group, Adv. Math. 28 (1978), 101-128

[15] V. Reiner, D. Stanton, V. Welker: The Charney-Davis quantity for certain graded posets, Sem. Comb. Loth. 50, 2003

[16] V. Schechtman, A. Varchenko: Arrangements of hyperplanes and Lie algebra homology, Invent. Math. 106 (1991), 139-194

[17] V. Schechtman, A. Varchenko: Quantum groups and homology of local systems, Algebraic Geometry and Anal. Geometry, ICM-90 Satellite Conference Proc., Tokyo, 1990, Springer, 1991, 182-191

[18] A. Varchenko: Combinatorics and topology of the disposition of affine hyperplanes in real space, Funct. Anal. Appl. 21 (1987), 9-19

[19] A. Varchenko: Bilinear Form of Real Configuration of Hyperplanes, Adv. Math. 97 (1993), 110-144

[20] A. Varchenko, I. Gel'fand: Heaviside Functions of a Configuration of Hyperplanes, Funct. Anal. Appl. 21 (1988), 255-270

[21] M. Wachs: Poset Topology:Tools and Applications, Geometric Combinatorics, Miller etal. eds., AMS, 2007

[22] D. Zagier: Realizability of a Model in Infinite States, Commun. Math. Phys. 147 (1992), 199-210

[23] G. M. Ziegler: Lectures on Polytopes, Springer, New York, 2006